Hydrogen atom

$$\phi \propto e^{-\alpha r}$$

$$k = \frac{1}{4\pi\epsilon_0}$$

$$H = \frac{-\hbar^2}{2\mu}\nabla^2 - \frac{ke^2}{r}$$

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) = \frac{1}{r^2}\left[2r\frac{d}{dr} + r^2\frac{d^2}{dr^2}\right]$$

$$= \boxed{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}}$$

$$\nabla^2\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}$$

$$\langle H \rangle = \int^{\infty} \phi^* \left[-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{ke^2}{r^2}\right)\phi \, \overset{4\pi r^2}{d\tau}$$

$$= \int_0 \left[-\frac{\hbar^2}{2\mu} \qquad r^2 \right]$$

$$\boxed{\int_0^\alpha r^n e^{-\alpha r} = \frac{n!}{\alpha^{n+1}}} \qquad \phi \propto e^{-\alpha r}$$

$$\langle \phi | \frac{d^2}{dr^2} | \phi \rangle = \int_0^\alpha e^{-\alpha r} \frac{d^2}{dr^2} \left(e^{-\alpha r}\right) \frac{4\pi r^2 dr}{d\tau}$$

$$= \int_0^\alpha e^{-\alpha r} \alpha^2 \, e^{-\alpha r} \, 4\pi r^2 dr$$

$$= 4\pi\alpha^2 \underline{\int_0^\alpha r^2 e^{-2\alpha r} dr}$$

$$= 4\pi\alpha^2 \times \left[\frac{2!}{(2\alpha)^3}\right] = \underline{\underline{\frac{\pi}{\alpha}}}$$

$$\langle\phi|\frac{2}{r}\frac{d}{dr}|\phi\rangle = 4\pi\int_0^{\infty} e^{-\alpha r}\times\frac{2}{r}\frac{\partial}{\partial r}\left(e^{-\alpha r}\right)\times r^2\,dr$$

$$= -8\pi\,\alpha\int_0^{\infty} r\,e^{-2\alpha r}\,dr$$

$$= -8\pi\alpha\left[\frac{1}{(2\alpha)^2}\right] = \frac{-8\pi}{4\alpha} = -\frac{2\pi}{\alpha}$$

$$\langle\phi|\frac{ke^2}{r}|\phi\rangle = 4\pi k e^2\int_0^{\infty}\frac{1}{r}\times r^2\,e^{-2\alpha r}\,dr$$

$$= 4\pi k e^2\left[\frac{1!}{(2\alpha)^2}\right]$$

$$= \frac{\pi k e^2}{\alpha 2}$$

$$\langle\phi|\phi\rangle = 4\pi\int_0^{\infty} r^2\,e^{-2\alpha r}\,dr$$

$$= 4\pi\times\ 2! \quad = \pi/$$

$$\overline{(2\alpha)^3} \quad /\alpha^3$$

$$\langle H \rangle = \frac{\int \phi^* H \phi \, d\tau}{\int_0^\infty \phi^* \phi \, d\tau} = \frac{-\frac{\hbar^2}{2M}\left[\frac{\pi}{\alpha} - \frac{2\pi}{\alpha}\right] - \frac{\pi k e^2}{\alpha^2}}{\pi/\alpha^3}$$

$$\langle H \rangle = \frac{\hbar^2 \alpha^2}{2M} - k e^2 \alpha \leftarrow$$

$$\frac{\partial \langle H \rangle}{\partial \alpha} = \frac{\hbar^2 \alpha}{M} - k e^2 = 0$$

$$\alpha = \frac{M k e^2}{\hbar^2}$$

$$\langle H \rangle = \frac{\hbar^2}{2M}\Big(\underbrace{\frac{M k e^2}{\hbar^2}}_{}\Big)^2 - \frac{M k e^2}{\hbar^2} k e^2$$

$\alpha^2 \downarrow$ $\downarrow \alpha$

$$\boxed{\langle H \rangle = -\frac{\mu k^2 e^4}{2\hbar^2}}$$