Chapter-1

Basic Laser Theory and Optical Resonators

1.1 Introduction

The name 'Laser' is an acronym for light amplification by stimulated emission of radiation. A laser is a device that produces an intense, concentrated and highly parallel beam of coherent light. Historically the laser is an outgrowth of maser (microwave amplification by stimulated emission of radiation), a similar device using microwaves instead of visible light.

The basic principle involved in the lasing action is the phenomenon of stimulated emission which was predicted by Einstein in 1917. The first successful laser was built by T H Maiman in 1960.

The three kinds of transitions involving electromagnetic radiation between two energy levels in an atom are,

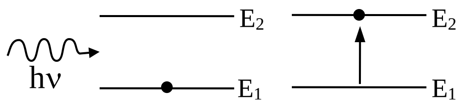


Fig.a: Induced absorption

1. Induced absorption

An atom which is initially in a lower state can go to the higher state by absorbing a photon of energy $E = E_2 - E_1 = h\nu$. This process is called induced absorption.

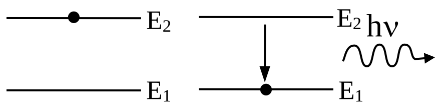


Fig.b: Spontaneous emission

2. Spontaneous emission

If the atom is initially in the higher state E_2, it can drop to the lower level by emitting a photon of energy $h\nu$. This is called the spontaneous emission.

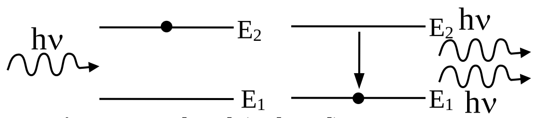


Fig.c: Stimulated (induced) emission

3. Induced (stimulated) emission

Einstein pointed out that a third possibility, called induced emission, in which an incident photon of energy $h\nu$, causes in an atom, a transition from a higher level to a lower level, producing two photons in coherence (i.e. in the same phase). Einstein showed that the induced emission has the same probability as the induced absorption. The rate of stimulated emission depends on the intensity of the external field and also on the number of atoms in the upper state.

In this chapter we mainly deal with the Einstein coefficients governing the above-mentioned processes, how light amplification takes place in presence of population inversion, the quantum theory for transition rates and the line broadening mechanisms.

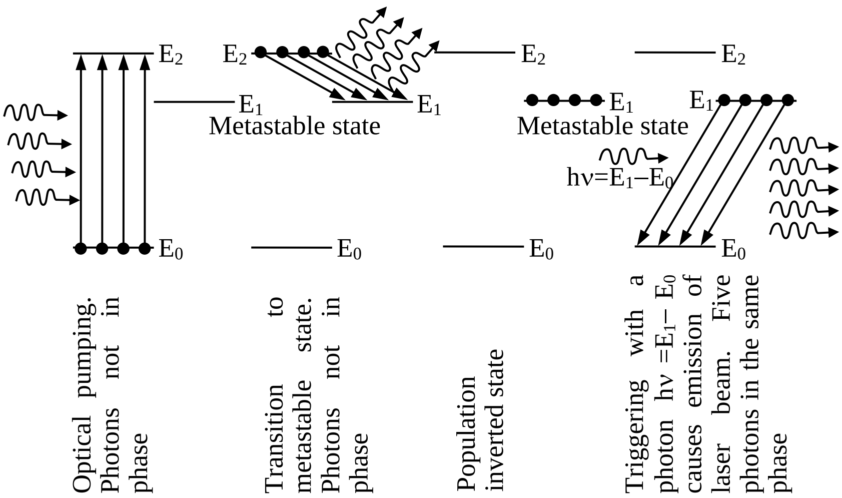
*Principle and requirements of a laser

The three main components of any laser device are the active medium, the pumping source and the optical resonator. The active medium consists of a collection of atoms, molecules or ions (in solid, liquid or gaseous form) which is capable of amplifying light waves. The laser may be a **three level laser** or a **four level laser**.

The simplest kind is a three level laser, which uses an assembly of atoms (or molecules) that have three states- a ground state, a metastable state and a higher excited state that can decay to the metastable state. For lasing action, we want more atoms in the higher energy state E_f than atoms in the lower state E_i. If this is achieved by some method and a photon of energy

$h\nu = E_f - E_i$ is passed through the assembly, there will be more induced emissions from the atoms in the higher level than induced absorptions by atoms in the lower level. The result will be an amplification of the incident original light. This is the concept that underlies the operation of a laser.

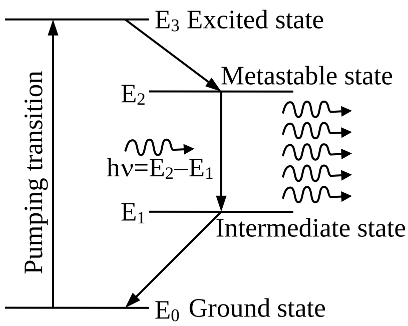
In a three level laser more than half the atoms must be in the metastable state for induced emission. In a **four level laser** there are four levels- excited state, metastable state, intermediate state and the ground state. The intermediate state is unstable. The laser transition is from metastable state to intermediate state. The population inversion is required between metastable state and intermediate state. Since the intermediate state is unstable and it decays rapidly to ground state the number of atoms in the intermediate state is very low. Hence even a moderate amount of pumping is enough to achieve population inversion.



Population inversion: Under ordinary conditions of thermal equilibrium the number of atoms in the higher levels is considerably smaller than the number of atoms in the lower energy states. For laser action, by some means, the atoms in the assembly are excited such that there is more number of atoms in the higher state E_f than in the lower state E_i. This is known as population inversion and is essential for laser action.

Optical pumping: This is a method to produce population inversion. In this method an external source of light is used to excite atoms in the ground state to higher state. Atoms first absorb photons from the external source and get excited to the higher states from which they finally decay into metastable state.

Metastable state: Laser action cannot occur if there are only two states. This is because the process of optical pumping induces transitions from ground state to the higher state as well as from the higher state to the lower state. When half the atoms are in each state, the rate of induced emission will be equal to the rate of induced absorption. So the assembly cannot ever have more than half its atoms in the higher state.



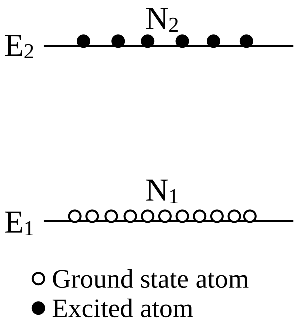
The lifetime of the excited atoms in the higher levels is of the order of nanoseconds. Therefore the population inversion is usually not possible in higher levels. In order the population inversion to takes place, the lifetime in a higher state is sufficiently large. Such long lived excited state is known as metastable state.

1.2 The Einstein's A&B coefficients

Consider a system of atoms having two energy states E_1 and E_2. Let N_1 and N_2 be the number of atoms per unit volume in the states 1 and 2 respectively. An atom which is initially in a lower state can go to the higher state by absorbing a photon of energy,

$$E = E_2 - E_1 = h\nu = \frac{h}{2\pi} 2\pi\nu = \hbar\omega$$

Or, $$\omega = \frac{E_2 - E_1}{\hbar} \qquad (1)$$



Now we define the energy density $u(\omega)$ such that $u(\omega)d\omega$ represents the radiation energy per unit volume within the frequency interval ω and $\omega + d\omega$. The rate of induced (stimulated) absorption per unit volume is proportional to the energy density $u(\omega)$ at a frequency ω of the radiation field due to the external photons and the number of atoms per unit volume in the lower state.

i.e. $$\left(\frac{dN_{12}}{dt}\right)_{\text{induced absorption}} \propto N_1 u(\omega)$$

i.e. Number of absorptions per unit time per unit volume, $\Gamma_{12} = B_{12} N_1 u(\omega)$ (2)

where, B_{12} is the coefficient of proportionality and is a characteristic of the energy levels. Now let us consider the transitions from higher level to lower level. Einstein postulated that these transitions are radiative. Atom goes to the lower level either through spontaneous emission or through induced (stimulated) emission. The spontaneous emission takes place in the absence of any external photon and is hence independent of the energy density $u(\omega)$ of the radiation field. So, the rate of spontaneous emission,

$$\left(\frac{dN_{21}}{dt}\right)_{\text{spontaneous emission}} \propto N_2$$

i.e. $$U_{21} = A_{21} N_2 \qquad (3)$$

But the induced emission depends on the energy density $u(\omega)$ of the radiation field also. Thus the rate of transition to the lower level,

$$\left(\frac{dN_{21}}{dt}\right)_{\text{induced emission}} \propto N_2 u(\omega)$$

i.e. Number of stimulated emissions per unit time per unit volume, $\Gamma_{21} = B_{21} N_2 u(\omega)$ (4)

The coefficients A_{21}, B_{21} and B_{12} are known as *Einstein's coefficients*. At thermal equilibrium the rate of upward transition is equal to the rate of net downward transition. That is,

$$B_{12} N_1 u(\omega) = A_{21} N_2 + B_{21} N_2 u(\omega)$$

i.e. $$u(\omega)[B_{12} N_1 - B_{21} N_2] = A_{21} N_2$$

$$u(\omega) = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} = \frac{A_{21}}{B_{12}\frac{N_1}{N_2} - B_{21}} \qquad (5)$$

According to Boltzmann's distribution formula,

$$\frac{N_1}{N_2} = e^{\frac{E_2 - E_1}{k_B T}} = e^{\frac{\hbar\omega}{k_B T}} \qquad (6)$$

where, k_B is the Boltzmann's constant and $\hbar\omega = h\nu = E_2 - E_1$.

Then eqn.5 becomes,

$$u(\omega) = \frac{A_{21}}{B_{12}e^{\frac{\hbar\omega}{k_B T}} - B_{21}} = \frac{1}{\frac{B_{12}}{A_{21}}e^{\frac{\hbar\omega}{k_B T}} - \frac{B_{21}}{A_{21}}} \quad (7)$$

Treating a system of photons as a gas obeying Bose-Einstein statistic, the Planck's law for energy density of radiation in a medium of refractive index μ_0 can be calculated as,

$$u(\omega) = \frac{\hbar\omega^3\mu_0^3}{\pi^2c^3}\left(\frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}\right) = \frac{1}{\frac{\pi^2c^3}{\hbar\omega^3\mu_0^3}e^{\frac{\hbar\omega}{k_B T}} - \frac{\pi^2c^3}{\hbar\omega^3\mu_0^3}} \quad (8)$$

[In a radiation chamber or cavity with free space $\mu_0 = 1$].

Comparing eqns.7 and 8 we get,

$$\frac{B_{12}}{A_{21}} = \frac{\pi^2c^3}{\hbar\omega^3\mu_0^3} \quad \text{or,} \quad \frac{A_{21}}{B_{12}} = \frac{\hbar\omega^3\mu_0^3}{\pi^2c^3} \quad (9a)$$

And

$$\frac{B_{21}}{A_{21}} = \frac{\pi^2c^3}{\hbar\omega^3\mu_0^3} \quad \text{or,} \quad \frac{A_{21}}{B_{21}} = \frac{\hbar\omega^3\mu_0^3}{\pi^2c^3} \quad (9b)$$

From eqns.9a and 9b, $B_{12} = B_{21} = B$. Also, we write $A_{21} = A$. (10)

Thus, the probabilities of stimulated absorption and stimulated emission are the same. The ratio between A and B coefficients is given by eqn.9. In the absence of stimulated emission the correct expression for $u(\omega)$ would not have been derived. So in order to obtain the correct form of $u(\omega)$, Einstein, in 1917, predicted the existence of stimulated emission.

At thermal equilibrium, using eqn.10 in eqns.7, we get,

$$u(\omega) = \frac{A_{21}}{B_{12}e^{\frac{\hbar\omega}{k_B T}} - B_{21}} = \frac{A}{B\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)}$$

i.e.

$$\frac{A}{Bu(\omega)} = e^{\frac{\hbar\omega}{k_B T}} - 1 \quad (11)$$

By eqn.11 it is clear that at thermal equilibrium at a temperature T, if $\omega << \frac{k_B T}{\hbar}$, the number of stimulated emissions (B) far exceeds the number of spontaneous emissions (A), while for $\omega >> \frac{k_B T}{\hbar}$ the number of spontaneous emissions far exceeds the number of stimulated emissions. For normal optical sources the temperature $T \sim 10^3$ K, then

$$\frac{k_B T}{\hbar} = \frac{1.38\times10^{-23}\text{J/K}\times10^3\text{K}}{1.054\times10^{-34}\text{J.sec}} = 1.31\times10^{14}\ \text{sec}^{-1}.$$

For optical region, [wavelength 400nm to 700 nm;

For 400 nm, $\omega = \frac{2\pi c}{\lambda} = \frac{2\times3.14\times3\times10^8}{400\times10^{-9}} = 4.71\times10^{15}$; $\frac{\omega}{k_B T/\hbar} = \frac{4.71\times10^{15}}{1.31\times10^{14}} = 36$]

$\omega \sim 4\times10^{15}$. That is, $\omega >> \frac{k_B T}{\hbar}$. Thus we find that *at optical frequencies the emission is predominantly due to the spontaneous transition and hence the emission from usual light sources is incoherent*.

By eqn.9b, $$B_{21} = \frac{\pi^2 c^3}{\hbar\omega^3} A_{21} = \frac{\pi^2 c^3}{\hbar\omega^3 t_{sp}} \quad (12)$$

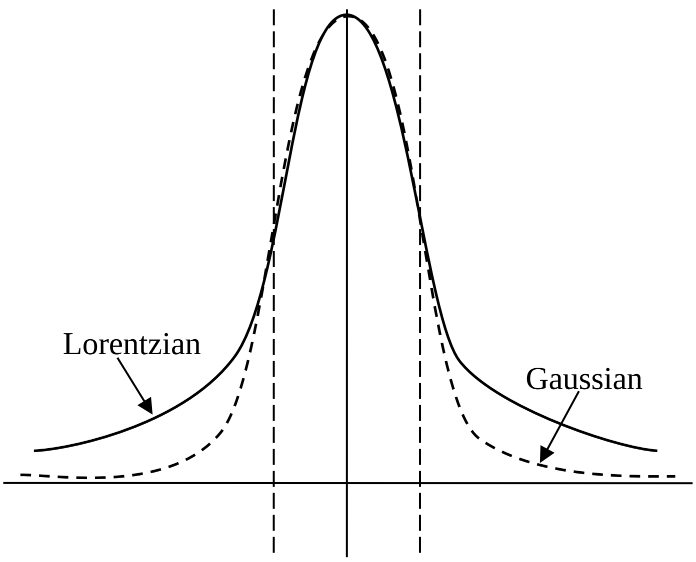
where, $t_{sp} = \frac{1}{A_{21}}$ represents the spontaneous lifetime of the upper level. (12a)

[The line-shape function *

The line-shape function is a real, nonnegative and usually normalized function. It is used for the mathematical description of the line shape for an absorptive transition. The transition may be electronic, rotational, or vibrational (i.e. visible, microwave or infrared radiation). **Spectral line shape** describes the form of a feature, observed in spectroscopy, corresponding to an energy change in an atom, molecule or ion. Ideal line shapes include **Lorentzian**, **Gaussian** and **Voigt functions**, whose parameters are the line position, maximum height and half-width. For each system the half-width of the shape function varies with temperature, pressure (or concentration) and phase.

The Lorentzian line shape function centered about any arbitrary frequency w_0, is given by,

$$L(\omega) = \frac{\Gamma}{\pi\left[\Gamma^2 + (\omega-\omega_0)^2\right]}$$



where Γ is the energy width. Note that the Lorentzian line shape function is a normalized function so that $\int_{-\infty}^{+\infty} L(\omega)\,d\omega = 1$.

A Gaussian function is also a useful line-shape function. Any source of inhomogneous broadening such as the Doppler shift or site differences of molecules in crystals or solution can be described as a Gaussian line-shape.

$$G(\omega) = \frac{1}{\Gamma\sqrt{\pi}} e^{-\frac{(\omega-\omega_0)^2}{\Gamma^2}}$$

The third line shape that has a theoretical basis is the Voigt function, which is a convolution of a Gaussian and a Lorentzian,

$$V(\omega) = \int_{-\infty}^{+\infty} G(\omega')\,L(\omega-\omega')\,d\omega'$$

The computation of a Voigt function and its derivatives is more complicated than a Gaussian or Lorentzian].

1.2.1 Further discussion of Einstein coefficients

In our discussion so far, we have assumed that the atom is capable of interacting with radiation of a particular frequency ω. In general, the atom can interact with radiation over a wide range of frequencies. The strength of interaction is a function of frequency, known as line-shape function. Let g(ω) represents the normalized line-function corresponding to the transition between levels 1 and 2. The function is usually normalized according to,

$$\int g(\omega)\,d\omega = 1 \qquad (13)$$

Then, the number of atoms per unit volume in level-1 capable of interacting with radiations of frequency range ω and ω+dω is

$$n_{1\omega}d\omega = N_1\, g(\omega)d\omega$$

and the number corresponding to level-2 is

$$n_{2\omega}d\omega = N_2\, g(\omega)d\omega$$

Now taken into account of the line-function, the number of stimulated emissions per unit time per unit volume is given by (modifying eqn.4),

$$\Gamma_{21} = N_2 \int B_{21} u(\omega) g(\omega) d\omega$$

Using eqn.12,

$$= N_2 \frac{\pi^2 c^3}{\hbar \mu_0^3 t_{sp}} \int \frac{u(\omega)}{\omega^3} g(\omega) d\omega \qquad (14)$$

Now we consider two specific cases.

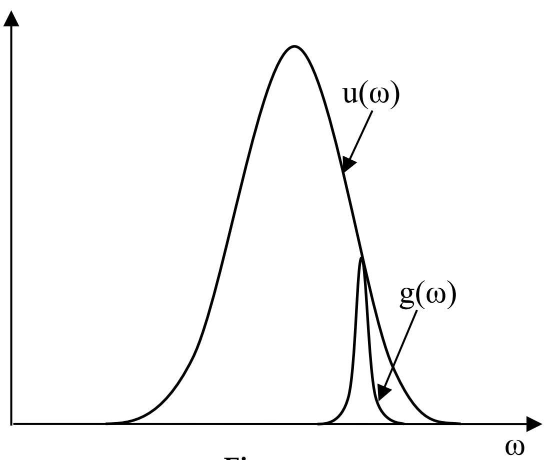


Fig.a

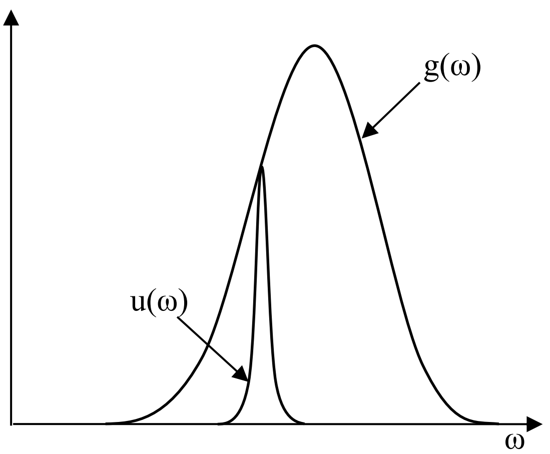


Fig.b

1. If the atoms are interacting with radiation whose spectrum is very broad compared to that of g(ω) as shown in fig.a, then we assume that over the region of integration where g(ω) is appreciable $\frac{u(\omega)}{\omega^3}$ is essentially constant and can be taken outside the integral in eqn.14. Then, using eqn.13, eqn.14 becomes,

$$\Gamma_{21} = N_2 \frac{\pi^2 c^3}{\hbar \mu_0^3 t_{sp}} \frac{u(\omega)}{\omega^3} \int g(\omega) d\omega = N_2 \frac{\pi^2 c^3}{\hbar \omega^3 \mu_0^3 t_{sp}} u(\omega) \qquad (15)$$

   where, ω now represents the transition frequency. Eqn.15 is consistent with eqn.4. Thus eqn.15 represents the rate of stimulated emission per unit volume when the atom interacts with the broad radiation.

2. In the other case, the atom is interacting with near monochromatic radiation. If the frequency of the incident radiation is ω', the u(ω) curve will be sharply peaked at $\omega = \omega'$ as compared to g(ω) as shown in fig.b. Then, we can take $\frac{g(\omega)}{\omega^3} = \frac{g(\omega)}{\omega'^3}$ outside the integral of eqn.14. Thus,

$$\Gamma_{21} = N_2 \frac{\pi^2 c^3}{\hbar \mu_0^3 t_{sp}} \frac{g(\omega')}{\omega'^3} \int u(\omega) d\omega = N_2 \frac{\pi^2 c^3}{\hbar \omega'^3 \mu_0^3 t_{sp}} g(\omega') U \qquad (16)$$

where, $$U = \int u(\omega) d\omega \qquad (17)$$

where, U is the energy density of the near monochromatic field. [U is the energy density by all frequencies, whereas u(ω) is the energy per unit volume per unit frequency range.

Similarly, for interaction with near monochromatic radiation, the number of stimulated absorptions per unit time per unit volume is given by (modifying eqn.2),

$$\Gamma_{12} = \frac{N_1 \pi^2 c^3}{\hbar \omega'^3 \mu_0^3 t_{sp}} g(\omega') U \qquad (18)$$

Absorption and emission cross sections: We know that the intensity and energy density of the electromagnetic wave are related by,

$$I = Uv,$$

where, $v = \frac{c}{\mu_0}$ is the wave velocity in the medium. Thus,

$$U = \frac{I}{v} = \frac{\mu_0 I}{c} \qquad (19)$$

If n is the number of photons crossing a unit area per unit time (also known as flux of photons) the intensity I is given by,

$$I = n\hbar\omega \qquad (20)$$

Using eqns.18 and 19, eqn.18 becomes, (since, $\omega = \omega'$),

$$\Gamma_{12} = \frac{N_1 \pi^2 c^3}{\hbar \omega'^3 \mu_0^3 t_{sp}} g(\omega') \frac{\mu_0 n \hbar \omega'}{c} = \frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega) N_1 n = \sigma_a N_1 n \qquad (21)$$

where, $$\sigma_a = \frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega) \qquad (22)$$

σ_a has the dimensions of area and is known as the absorption cross section. Similarly, eqn.15 can be written as,

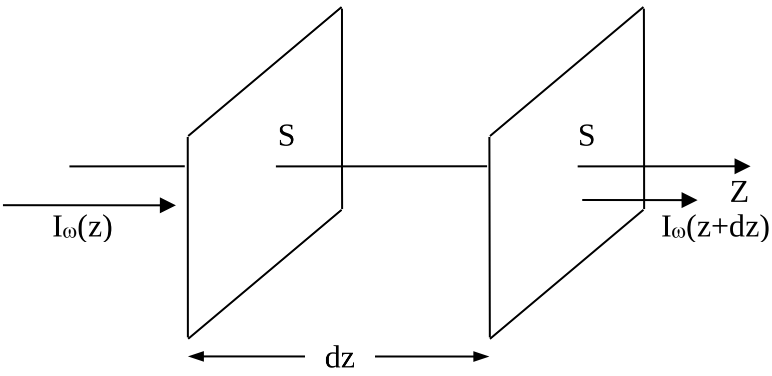
$$\Gamma_{21} = N_2 \frac{\pi^2 c^3}{\hbar \omega^3 \mu_0^3 t_{sp}} u(\omega) = N_2 \frac{\pi^2 c^3}{\hbar \omega^3 \mu_0^3 t_{sp}} \frac{\mu_0 n \hbar \omega}{c}$$

$$= \frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega) N_2 n = \sigma_e N_2 n \qquad (23)$$

where, $$\sigma_e = \frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega) \qquad (24)$$

Eqns.22. and 24 show that the absorption and emission cross sections are equal and they are functions of frequency ω. They are related to the line broadening function $g(\omega)$ and the lifetime t_{sp}.

1.3 Light amplification



Consider a collection of atoms of a particular medium. A near monochromatic beam of frequency ω is allowed to propagate in the Z-direction through the medium. In order to obtain the expression for the rate of change of intensity of the beam as it propagates through the medium, we imagine two parallel plane surfaces of equal area S such that the surfaces are perpendicular to the Z-axis, which passes through their centers. Let dz be the separation between the two planes.

Volume of the medium between the planes $= Sdz$

Number of stimulated absorptions per unit time $= \Gamma_{12}\, Sdz$

Energy absorbed per unit time in the volume element Sdz $= \Gamma_{12}\, Sdz\, \hbar\omega$

Energy gain by stimulated emission $= \Gamma_{21}\, Sdz\, \hbar\omega$

We can neglect the gain in energy due to spontaneous emissions because the radiations arising out of it propagate in random directions. We assume that half of them propagate in the positive z-direction and remaining half in the negative z-direction. Thus,

Net amount of energy absorbed per unit time in the volume element Sdz and in the frequency interval ω and $\omega + d\omega$ $= \left(\Gamma_{12} - \Gamma_{21}\right)\hbar\omega\, Sdz$

Let $I_\omega(z)$ and $I_\omega(z+dz)$, respectively, be the intensities of the radiation entering the volume element and that leaving it. Then,

Energy entering into the volume element per unit time $= I_\omega(z)\, S$

Energy leaving the volume element per unit time $= I_\omega(z+dz)\, S$

Assuming the change in energy of the radiation is linear as it propagates in the Z-direction, we can write,

$$I_\omega(z+dz)S = I_\omega S + \frac{\partial I_\omega}{\partial z} Sdz$$

Energy leaving the volume element per unit time $= I_\omega S + \dfrac{\partial I_\omega}{\partial z} Sdz$ (25)

$\therefore$ Net amount of energy leaving the volume element per unit time $= \dfrac{\partial I_\omega}{\partial z} Sdz$

This must be equal to the negative of the net energy absorbed by the medium in between z and z+dz per unit time. Thus,

$$\frac{\partial I_\omega}{\partial z} Sdz = -\left(\Gamma_{12} - \Gamma_{21}\right)\hbar\omega\, Sdz$$

Using eqn.16 and 18 (with $\omega' = \omega$ and energy density U),

$$\frac{\partial I_\omega}{\partial z} Sdz = -\left(\frac{N_1 \pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g(\omega) U - N_2 \frac{\pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g(\omega) U\right)\hbar\omega\, Sdz$$

$$= -\frac{\pi^2 c^3}{\omega^2 \mu_0^3 t_{sp}} g(\omega) U (N_1 - N_2) S dz$$

Or, $$\frac{\partial I_\omega}{\partial z} = -\frac{\pi^2 c^3}{\omega^2 \mu_0^3 t_{sp}} g(\omega) U (N_1 - N_2) \quad (26)$$

The energy density U and the intensity I_ω are related through the equation,

$$I_\omega = vU = \frac{c}{\mu_0} U \quad (27)$$

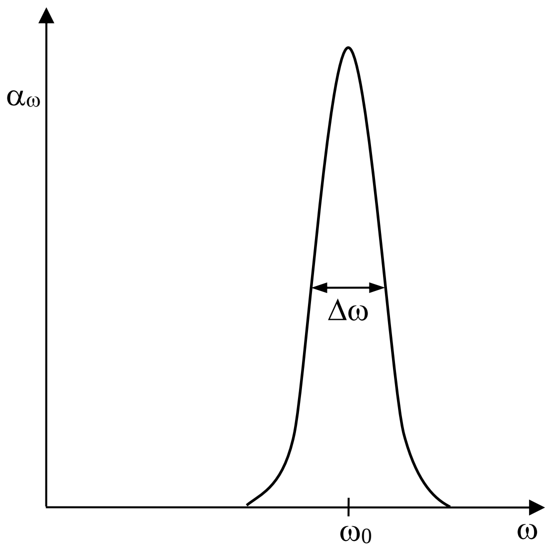
where, v is the velocity of propagation of the wave through the medium and μ_0 is the refractive index of the medium. Then eqn.26 becomes,

$$\frac{\partial I_\omega}{\partial z} = -\frac{\pi^2 c^3}{\omega^2 \mu_0^3 t_{sp}} g(\omega) \frac{\mu_0 I_\omega}{c} (N_1 - N_2)$$

Or, $$\frac{dI_\omega}{I_\omega} = -\frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega) (N_1 - N_2) dz = -\alpha_\omega dz \quad (28)$$

where, $$\alpha_\omega = \frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega) (N_1 - N_2) = -\gamma \quad (29)$$

Figure below is a typical plot of α_ω with ω.



Integrating eqn.28, we get,

$$\ln I_\omega(z) = -\alpha_\omega z + C$$

When z = 0, $I_\omega(z) = I_\omega(0)$.

Then, $C = \ln I_\omega(0)$. Thus, we get,

$$\ln I_\omega(z) = -\alpha_\omega z + \ln I_\omega(0)$$

i.e. $$\ln\left(\frac{I_\omega(z)}{I_\omega(0)}\right) = -\alpha_\omega z$$

Taking exponential and rearranging, we get,

$$I_\omega(z) = I_\omega(0) e^{-\alpha_\omega z} \quad (30)$$

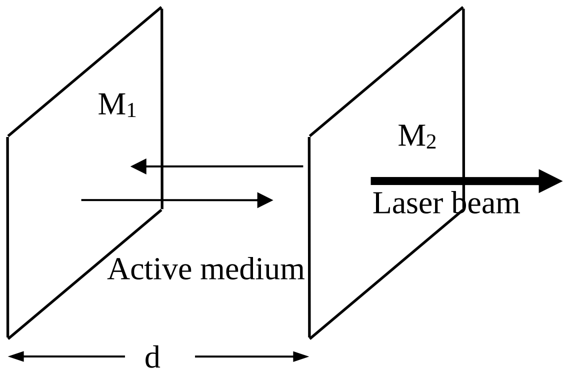
Now we consider the two cases.

Case (a): $N_1 > N_2$. In this case α_ω is positive. Then by eqn.30 it is clear that the intensity of the beam decreases exponentially. *Hence at thermal equilibrium, if the number of atoms in the lower level is greater than the number of atoms in the higher level, the energy of the beam decreases exponentially as it propagates through the medium.*

Case (b) : $N_2 > N_1$. This case is known as *population inversion*. In this case α_ω is negative and the intensity increases exponentially. That is, *at thermal equilibrium if there are more atoms in the excited states than the atoms in the lower level, the intensity of the beam increases exponentially as it propagates through the medium*. This is known as *light amplification*.

1.3.1 Threshold conditions for laser action

In an actual laser system, the active medium, which produces the light amplification, is placed in between two parallel mirrors facing each other. This arrangement is known as an optical resonator. The region between the mirrors is known as cavity. In order to produce laser beam the oscillations must be sustained in the cavity. This is possible only if the net losses suffered by the beam must be compensated by the gain of the medium. The threshold and under steady state operation of the laser system they are exactly equal.



Let 'd' be the length of the active medium. Let R_1 and R_2, respectively, are the reflectivities of the mirrors M_1 and M_2 at the two ends of the laser resonator. The intensity of the beam at one of the mirrors, say M_2 is represented by I. While travelling through the active medium the beam gets gain in energy due to light amplification or suffers loss in energy due to absorption, scattering etc. in the laser medium. By eqn.23 the beam gets amplified by the factor $e^{-\alpha_\omega d}$. The diminishing of the beam depends on the passive parameters of the medium. The beam gets diminished by the factor $e^{-\alpha_c d}$. Here, α_c represents the average loss per unit length due to all loss mechanisms (other than the finite reflectivity) such as scattering loss, diffraction loss due to finite mirror size, etc. Now we use $\alpha_\omega = -\gamma$. Then,

Intensity of the beam when it reaches the mirror M_1 after travelling a distance 'd' in the laser medium, $I' = Ie^{(\gamma-\alpha_c)d}$

Intensity of the beam after reflection at the mirror M_1, $I_1 = I'R_1 = IR_1e^{\gamma d}e^{-\alpha_c d}$

$= IR_1e^{(\gamma-\alpha_c)d}$

Intensity of the beam when it reaches the mirror M_2 after traversing a further distance 'd' in the laser medium, $I'' = I_1e^{(\gamma-\alpha_c)d} = IR_1e^{(\gamma-\alpha_c)d}e^{(\gamma-\alpha_c)d} = IR_1e^{2(\gamma-\alpha_c)d}$

Intensity of the beam after reflection at the mirror M_2, $I_2 = I''R_2 = IR_1R_2e^{2(\gamma-\alpha_c)d}$

The laser action takes place only if $I_2 \geq I$ or, $\frac{I_2}{I} \geq 1$. From the above equation we get, the **condition for the laser oscillation to begin** is $R_1R_2e^{2(\gamma-\alpha_c)d} \geq 1$ (31)

The equality sign would correspond to the threshold value for oscillation. Remember for light amplification population inversion is needed. Eqn.31 can be written as,

Condition for laser oscillation is, $e^{2\gamma d} \geq \frac{1}{R_1R_2}e^{2\alpha_c d}$

Taking exponential of eqn.32, we get,

$$2\gamma d \geq 2\alpha_c d - \ln R_1R_2$$

i.e.
$$\gamma \geq \alpha_c - \frac{1}{2d}\ln R_1R_2 \quad (32)$$

[The RHS, which depends on the passive cavity parameters, is related to the quality factor Q of the passive resonator. Later we will show that

$$2\alpha_c d = \ln R_1 R_2 + \frac{4\pi\nu\mu_0 d}{cQ}$$

Or, $$\ln R_1 R_2 = 2\alpha_c d - \frac{4\pi\nu\mu_0 d}{cQ} = 2\alpha_c d - \frac{2\mu_0 d}{ct_c} \qquad (33)$$

where, $t_c = \dfrac{Q}{\omega} = \dfrac{Q}{2\pi\nu}$ is the *passive cavity lifetime of the resonator*. (34)

Then, by eqn.33,

$$\frac{1}{t_c} = \frac{c}{2\mu_0 d}\left(2\alpha_c d - \ln R_1 R_2\right) \qquad] \qquad (35a)$$

Or, $$\alpha_c - \frac{1}{2d}\ln R_1 R_2 = \frac{\mu_0}{ct_c} \qquad (35b)$$

Using eqn.29 in eqn.32, we get,

$$\frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega)\left(N_2 - N_1\right) \geq \alpha_c - \frac{1}{2d}\ln R_1 R_2$$

Using eqn.35b,

$$\frac{\pi^2 c^2}{\omega^2 \mu_0^2 t_{sp}} g(\omega)\left(N_2 - N_1\right) \geq \frac{\mu_0}{ct_c}$$

i.e. $$N_2 - N_1 \geq \frac{\mu_0^3 \omega^2}{\pi^2 c^3}\left(\frac{t_{sp}}{t_c}\right)\frac{1}{g(\omega)} = \frac{\mu_0^3 \omega^3}{\pi^2 c^3}\left(\frac{t_{sp}}{Q}\right)\frac{1}{g(\omega)} \qquad (36a)$$

Or, $$\geq \frac{4\mu_0^3 \nu^2}{c^3}\left(\frac{t_{sp}}{t_c}\right)\frac{1}{g(\omega)} \qquad (36b)$$

where, ν is the oscillation frequency at the centre of the resonator mode, μ_0 is the refractive index of the medium and c is the velocity of the electromagnetic wave in free space. Eqn.36 gives the threshold population inversion required for the laser action. The minimum threshold value (corresponding to the equality) correspond to centre of the line where $g(\omega)$ is maximum.

The method to produce population inversion is known as *optical pumping*. As the laser medium is pumped harder and harder, the population inversion between the two levels goes on increasing. The mode that lies nearest to the resonance frequency of the atomic system reaches threshold first and begins to oscillate. As the pumping is still further increased the nearby modes may also reach threshold and start oscillating.

By eqn.36, we can state the following conditions required to have the low threshold value of population inversion.

1. The value of, $t_c = \dfrac{Q}{2\pi\nu}$ should be large. That is, Q must be large or cavity losses must be small.
2. The value of $g(\omega)$ at the centre of line should be large. For a Lorentzian line $g(\omega) = \dfrac{2}{\pi\,\Delta\omega}$ and for a Gaussian line $g(\omega) = \dfrac{2(\pi \ln 2)^{1/2}}{\pi\,\Delta\omega}$. Thus, smaller values of line width $\Delta\omega$ lead to smaller values of threshold population inversion.
3. Small values of t_{sp} lead small values of threshold population inversion. That is the relaxation times of transitions corresponding to spontaneous emission should be short. In general, population inversion is more easily obtained on transitions which have longer relaxation times.

4. By eqn.36a it is clear that the value of threshold population inversion is approximately (frequency dependence of other terms should be considered) proportional to ω^3. Hence it is much easier to obtain laser action at longer wavelengths (infrared region) as compared to the shorter wavelengths (ultraviolet region).

Ruby laser as an example: In order to get an idea of the magnitude of population inversion required for laser action we consider ruby laser. Ruby is a crystal of aluminum oxide Al_2O_3 (corundum) doped with approximately 0.05 percent of chromium ions in the form of Cr_2O_3, so that some Al atoms in the crystal lattice are replaced by Cr^{3+} ions. We consider the laser to be oscillating at the frequency corresponding to the peak of the emission line.

Population of Cr^{3+} ions, per cm^3, $N = N_1 + N_2 = 1.6\times10^9$

Value of $g(\omega)$ at the peak of the line, $= \dfrac{2}{\pi\,\Delta\omega}$ (Lorentzian line shape)

$$\therefore \text{ Threshold population inversion density, } N_2 - N_1 = \frac{4\nu^2}{c^3 g(\omega)}\frac{t_{sp}}{t_c} = \frac{4\nu^2}{c^3 \dfrac{2}{\pi\,\Delta\omega}}\frac{t_{sp}}{t_c}$$

$$= \frac{4\pi^2\nu^3}{c^3}\frac{\Delta\nu}{\nu}\frac{t_{sp}}{t_c} = \frac{4\pi^2}{\lambda^3}\frac{\Delta\nu}{\nu}\frac{t_{sp}}{t_c} \qquad (37)$$

[For Gaussian line shape $\Delta\nu$ in eqn.37 must be replaced by $\dfrac{\Delta\nu}{(\pi \ln 2)^{1/2}}$]

For ruby laser transition

Wavelength of ruby laser, $\lambda_{\text{free space}}$ = 694.3 nm

Refractive index of ruby rod n_0 = 1.76

$$\therefore \text{ Wavelength of light in the ruby rod } \lambda = \frac{\lambda_{\text{free space}}}{n_0} = \frac{694.3\times10^{-7}}{1.76} \approx 4\times10^{-5}\,\text{cm}$$

$$\nu = \frac{c}{\lambda_{\text{free space}}} = \frac{3\times10^{10}}{694.3\times10^{-7}} \approx 4.3\times10^{14}\,\text{sec}^{-1}.$$

Since frequency of light in the medium is same as the frequency in the free space. But wavelengths are different.

$$\Delta\nu \approx 1.5\times10^{11}\,\text{sec}^{-1}.$$

$$t_{sp} \approx 3\times10^{-3}\ \text{sec} \qquad (38)$$

If 'd' is the length of the optical cavity, n_0 is the refractive index of the medium filling the cavity and 'x' is the fractional loss per round trip it can be shown that (later we see it),

$$t_c = \frac{2n_0 d}{c\,\ln\left(\dfrac{1}{1-x}\right)} \qquad (39)$$

If length of the cavity is 5 cm and x = 10%, then

$$t_c = \frac{2\times1.76\times5}{3\times10^{10}\,\ln\left(\dfrac{1}{1-0.1}\right)} \approx 6\times10^{-9}\ \text{sec} \qquad (40)$$

$$\text{Then, threshold population density in ruby laser, } N_2 - N_1 = \frac{4\pi^2}{\lambda^3}\frac{\Delta\nu}{\nu}\frac{t_{sp}}{t_c}$$

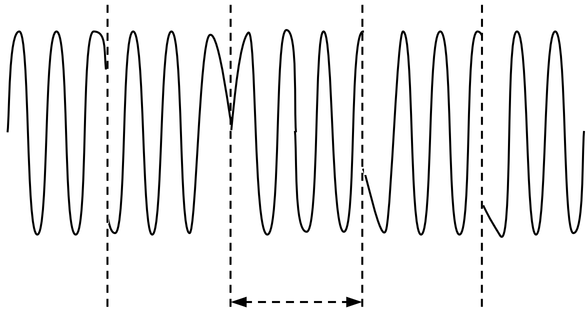
$$= \frac{4\times3.14^2}{(4\times10^{-5})^3}\times\frac{1.5\times10^{11}}{4.3\times10^{14}}\times\frac{3\times10^{-3}}{6\times10^{-9}} \approx 1.1\times10^{16}\ Cr^{3+}\ \text{ions/cm}^3.$$

1.4 Line broadening mechanisms

The radiation coming out of a collection of atoms making transitions between two energy levels is never perfectly monochromatic. Thus, there is broadening of spectral lines. This line broadening is described in terms of line shape function $g(\omega)$. Now we discuss some important line-broadening mechanisms and the corresponding line shape functions. A study of this is of great importance as it determines the operation characteristics of the laser, e.g. the threshold population inversion, the number of oscillating modes etc. There are three types of line broadening mechanisms and a convolution of all these three mechanisms.

(1) **Collision/pressure broadening**: This is due to the finite lifetime in quantum states owing to collision.

We first consider the line-shape function corresponding to the collisions that occur in a collection of atoms in the gaseous form. There are random collisions between atoms. Thus, an atom when interacting with the incident electromagnetic wave sees a field which changes its phase abruptly at each collision. Thus, the atom no longer sees a monochromatic wave but instead a wave like that shown in the figure. If τ is the average time between two collisions, there is abrupt change in phase of the wave at time intervals τ as shown in the figure. Thus, in this case the line-shape function would be given by (apart from some proportionality constant) the power spectrum of the field shown in the figure. The field of this type can be written in the form,



$$E(t) = E_0 e^{i(\omega_0 t+\phi)} \qquad (1)$$

where the phase constant ϕ remains constant for $t_0 \le t \le t_0 + \tau$ (2)

At each collision the phase ϕ changes abruptly. The frequency spread of such a wave is obtained by a Fourier transform as given below.

$$E(\omega) = \frac{1}{2\pi}\int_{t_0}^{t_0+\tau} E_0 e^{i(\omega_0 t+\phi)} e^{-i\omega t} dt = \frac{1}{2\pi}\int_{t_0}^{t_0+\tau} E_0 e^{i\{(\omega_0-\omega)t+\phi\}} dt$$

Here ω is the radiation (absorption or emission) frequency.

Put $i\{(\omega_0-\omega)t+\phi\} = x$

i.e. $i(\omega_0-\omega)dt = dx$

Then,

$$E(\omega) = \frac{E_0}{2\pi i(\omega_0-\omega)}\int_{t_0}^{t_0+\tau} e^x dx = \frac{E_0}{2\pi i(\omega_0-\omega)}\left[e^x\right]_{t_0}^{t_0+\tau}$$

$$= \frac{E_0}{2\pi i(\omega_0-\omega)}\left[e^{i\{(\omega_0-\omega)t+\phi\}}\right]_{t_0}^{t_0+\tau}$$

$$= \frac{E_0}{2\pi i(\omega_0-\omega)}\left[e^{i\{(\omega_0-\omega)(t_0+\tau)+\phi\}} - e^{i\{(\omega_0-\omega)t_0+\phi\}}\right]$$

$$= \frac{E_0}{2\pi i(\omega_0-\omega)}\left[e^{i\{(\omega_0-\omega)t_0+(\omega_0-\omega)\tau+\phi\}} - e^{i\{(\omega_0-\omega)t_0+\phi\}}\right]$$

$$= \frac{E_0}{2\pi i(\omega_0-\omega)} e^{i\{(\omega_0-\omega)t_0+\phi\}}\left[e^{i(\omega_0-\omega)\tau} - 1\right]$$

Thus, the frequency distribution of intensity (power spectrum) such a wave is given by,

$$I(\omega) \propto \left|E(\omega)\right|^2 = E^*(\omega)E(\omega)$$

$$\propto -\frac{E_0}{2\pi i(\omega_0-\omega)}e^{-i\{(\omega_0-\omega)t_0+\phi\}}\left[e^{-i(\omega_0-\omega)\tau}-1\right]\frac{E_0}{2\pi i(\omega_0-\omega)}e^{i\{(\omega_0-\omega)t_0+\phi\}}\left[e^{i(\omega_0-\omega)\tau}-1\right]$$

$$\propto \frac{E_0^2}{4\pi^2(\omega_0-\omega)^2}\left[e^{-i(\omega_0-\omega)\tau}-1\right]\left[e^{i(\omega_0-\omega)\tau}-1\right]$$

$$\propto \frac{E_0^2}{4\pi^2(\omega_0-\omega)^2}\left[1-e^{-i(\omega_0-\omega)\tau}-e^{i(\omega_0-\omega)\tau}+1\right]$$

$$\propto \frac{E_0^2}{4\pi^2(\omega_0-\omega)^2}\left[2-\cos(\omega_0-\omega)\tau+i\sin(\omega_0-\omega)\tau-\cos(\omega_0-\omega)\tau-i\sin(\omega_0-\omega)\tau\right]$$

$$\propto \frac{E_0^2}{4\pi^2(\omega_0-\omega)^2}\left[2-2\cos(\omega_0-\omega)\tau\right] = \frac{E_0^2}{4\pi^2(\omega-\omega_0)^2}\left[2-2\cos(\omega-\omega_0)\tau\right]$$

$$\propto \frac{E_0^2}{4\pi^2(\omega-\omega_0)^2}\left[2-2\left\{1-2\sin^2\left(\frac{\omega-\omega_0}{2}\right)\tau\right\}\right]$$

$$\propto \frac{E_0^2}{\pi^2(\omega-\omega_0)^2}\sin^2\left(\frac{\omega-\omega_0}{2}\right)\tau \tag{3}$$

Let $P(\tau)d\tau$ represents the probability that the atom suffers a collision after a time interval between τ and $\tau + d\tau$. By kinetic theory of gases $P(\tau)d\tau$ is given by,

$$P(\tau)d\tau = \frac{1}{\tau_0}e^{-\frac{\tau}{\tau_0}}d\tau \tag{4}$$

with, $$\int_0^\infty P(\tau)d\tau = 1 \tag{5}$$

and $$\int_0^\infty \tau P(\tau)d\tau = \tau_0 \tag{6}$$

At any instant the radiation is from atoms that have different values of τ. Thus, in order to obtain the spectral density we must multiply $I(\omega)$ by $P(\tau)d\tau$ and integrate from 0 to ∞. Thus, the frequency distribution of the radiation causing the transition is given by,

$$g(\omega) \propto \int_0^\infty I(\omega)P(\tau)d\tau$$

$$\propto \frac{E_0^2}{\pi^2(\omega-\omega_0)^2}\frac{1}{\tau_0}\int_0^\infty e^{-\frac{\tau}{\tau_0}}\sin^2\left(\frac{\omega-\omega_0}{2}\right)\tau d\tau$$

$$= \frac{KE_0^2}{\pi^2(\omega-\omega_0)^2}\frac{1}{\tau_0}\int_0^\infty e^{-\frac{\tau}{\tau_0}}\sin^2\left(\frac{\omega-\omega_0}{2}\right)\tau d\tau,$$

where, K is the proportionality constant. Put $U = \sin^2\left(\frac{\omega-\omega_0}{2}\right)\tau$ and $dV = e^{-\frac{\tau}{\tau_0}}d\tau$, then integrate using $\int UdV = UV - \int VdU$. Applying the limits first term reduces to zero.

Then using formulae, 2 sinA cosA = sin2A and $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$,
we get,

$$\int_0^\infty e^{-\frac{\tau}{\tau_0}}\sin^2\left(\frac{\omega-\omega_0}{2}\right)\tau d\tau = \frac{\tau_0}{2}\left[\frac{(\omega-\omega_0)^2}{\frac{1}{\tau_0^2}+(\omega-\omega_0)^2}\right]$$

Then, $$g(\omega) = \frac{KE_0^2}{\pi^2(\omega-\omega_0)^2}\frac{1}{\tau_0}\frac{\tau_0}{2}\left[\frac{(\omega-\omega_0)^2}{\frac{1}{\tau_0^2}+(\omega-\omega_0)^2}\right] = \frac{KE_0^2\tau_0^2}{2\pi^2}\left[\frac{1}{1+(\omega-\omega_0)^2\tau_0^2}\right]$$

The proportionality constant and hence the constant factor is adjusted such that $\int g(\omega)d\omega = 1$. Thus, we get,

$$g(\omega) = \frac{\tau_0}{\pi}\left[\frac{1}{1+(\omega-\omega_0)^2\tau_0^2}\right] = \frac{\tau_0}{\pi}\left[\frac{1}{1+(\omega_0-\omega)^2\tau_0^2}\right] \quad (7)$$

The distribution g(ω) given by eqn.7 is known as a *Lorentzian* and is plotted in the figure below (page 28). The peak of g(ω) can be determined by the condition $\frac{dg}{d\omega}=0$. The peak lies at $\omega=\omega_0$ and has a value, $\frac{\tau_0}{\pi}$. We can also find out the width at the half maximum, $\frac{\tau_0}{2\pi}$. That is,

$$\frac{\tau_0}{2\pi} = \frac{\tau_0}{\pi}\left[\frac{1}{1+(\omega_0-\omega)^2\tau_0^2}\right]$$

$$(\omega_0-\omega)^2 = \frac{1}{\tau_0^2}$$

$$\therefore \quad (\omega_0-\omega) \text{ or } (\omega-\omega_0) = \frac{1}{\tau_0}$$

Full width at half maximum, $$\Delta\omega = \frac{2}{\tau_0} \quad (8)$$

The mean time between collisions depends on the mean free path and average speed of atoms in the gas. Hence, they depend upon pressure, temperature and mass of atoms. The approximate expression for average collision time for monatomic gas is,

$$\tau_0 = \frac{1}{8\pi pa^2}\left(\frac{2}{3}Mk_BT\right)^{1/2}$$

where, p is the pressure of the gas, a is the atomic radius, M is the atomic mass, k_B is the Boltzmann's constant and T is the temperature.

2. Doppler broadening: This is due to thermal motion of atoms. Now we calculate the effect of the thermal motions of the gas atoms. According to kinetic theory, gas atoms undergo random motions. When such a moving atom interacts with radiation, the apparent frequency of the incident wave is different from the frequency seen from a stationary atom. This is called Doppler Effect. Due to this effect there is shifting of the resonance frequency of the atom. Let ω be the frequency of the incident wave. Also, we assume that the wave is travelling along the Z-axis. If

v_z is the component of velocity of the atom in the z-direction, the relative velocity of the wave with respect to the atom is $c - v_z$. Then the

Apparent frequency, $$\nu' = \frac{c - v_z}{\lambda}$$

Original frequency, $$\nu = \frac{c}{\lambda}$$

$$\frac{\nu'}{\nu} = \frac{c - v_z}{c} = 1 - \frac{v_z}{c}$$

Or, $$\nu' = \nu\left(1 - \frac{v_z}{c}\right)$$

i.e. $$2\pi\nu' = 2\pi\nu\left(1 - \frac{v_z}{c}\right)$$

Thus, the apparent angular frequency as seen by the atom is given by,

$$\omega' = \omega\left(1 - \frac{v_z}{c}\right) \tag{9a}$$

Or, $$\omega = \omega'\left(1 - \frac{v_z}{c}\right)^{-1} \approx \omega'\left(1 + \frac{v_z}{c}\right) \tag{9b}$$

Let ω_0 be the resonant frequency (transition frequency) of the atom. ($\omega_{21} = \omega'$). In order that the incident radiation may interact strongly with the atom, the resonant frequency of the atom must be equal to the apparent frequency of the wave, i.e. $\omega_0 = \omega'$. Then by eqn.9b,

$$\omega \approx \omega_0\left(1 + \frac{v_z}{c}\right) \tag{10}$$

Thus, the effect of motion of the atom is to change the resonant frequency of the atom.

According to the Maxwell-Boltzmann distribution, the probability of an atom having z-component of velocity lying between v_z and $v_z + dv_z$ is given by,

$$P(v_z)dv_z = \left(\frac{M}{2\pi k_B T}\right)^{1/2} e^{-\left(\frac{Mv_z^2}{2k_B T}\right)} dv_z \tag{11}$$

where, M is the mass of the atom, T is the temperature of the gas and k_B is the Boltzmann's constant.

By eqn.10, $$\frac{\omega}{\omega_0} = 1 + \frac{v_z}{c}$$

i.e. $$\frac{v_z}{c} = \frac{\omega}{\omega_0} - 1 = \frac{\omega - \omega_0}{\omega_0}$$

Or, $$v_z = \left(\frac{\omega - \omega_0}{\omega_0}\right)c \tag{12a}$$

and, $$dv_z = \frac{c}{\omega_0}d\omega \tag{12b}$$

The probability $g(\omega)d\omega$ that the transition frequency lies between ω and $\omega + d\omega$ is same as the probability that the z-component of velocity lies between v_z and $v_z + dv_z$. Using eqns.12a and b in eqn.11 we get,

$$g(\omega)d\omega = \frac{c}{\omega_0}\left(\frac{M}{2\pi k_B T}\right)^{1/2} e^{-\left(\frac{Mc^2(\omega-\omega_0)^2}{2k_B T\omega_0^2}\right)}d\omega = \left(\frac{Mc^2}{2\pi k_B T\omega_0^2}\right)^{1/2} e^{-\left(\frac{Mc^2(\omega-\omega_0)^2}{2k_B T\omega_0^2}\right)}d\omega \quad (13)$$

This corresponds to the Gaussian distribution. The distribution curve is plotted in the figure below (page 19). At the peak $\frac{dg}{d\omega}=0$. Then we get the peak is obtained for $\omega=\omega_0$. For this frequency, the maximum value of g(ω) is $\left(\frac{Mc^2}{2\pi k_B T\omega_0^2}\right)^{1/2}$. For half maximum,

$$\left(\frac{Mc^2}{2\pi k_B T\omega_0^2}\right)^{1/2}\frac{1}{2} = \left(\frac{Mc^2}{2\pi k_B T\omega_0^2}\right)^{1/2} e^{-\left(\frac{Mc^2(\omega-\omega_0)^2}{2k_B T\omega_0^2}\right)}$$

$$\ln 2 = \frac{Mc^2(\omega-\omega_0)^2}{2k_B T\omega_0^2}$$

$$\omega-\omega_0 = \omega_0\left(\frac{2k_B T\ln 2}{Mc^2}\right)^{1/2}$$

$$\therefore \text{ Full width at half maximum } \Delta\omega = 2(\omega-\omega_0) = 2\omega_0\left(\frac{2k_B T\ln 2}{Mc^2}\right)^{1/2} \quad (14)$$

$$\text{Peak value of } g(\omega) = \left(\frac{Mc^2}{2\pi k_B T\omega_0^2}\right)^{1/2} = \frac{2}{\Delta\omega}\left(\frac{\ln 2}{\pi}\right)^{1/2}$$

Thus, in terms of $\Delta\omega$, the Gaussian line-shape function can be written as,

$$g(\omega) = \frac{2}{\Delta\omega}\left(\frac{\ln 2}{\pi}\right)^{1/2} e^{-4(\ln 2)\frac{(\omega-\omega_0)^2}{(\Delta\omega)^2}} \quad (15)$$

3. Natural broadening: This is the inherent line width as a result of the finite lifetime of the excited states corresponding to the spontaneous emission. The rate of transition from the level 2 to 1corresponding to spontaneous emission is given by eqn.3 sec.1.2.

$$\left(\frac{dN_2}{dt}\right)_{\text{spontaneous emission}} = -A_{21}N_2$$

Rearranging and integrating, we get,

$$N_2 = N_{20}e^{-A_{21}t} \quad (16)$$

The energy of the photon emitted is given by,

$$\hbar\omega_0 = E_2 - E_1$$

Then the energy emitted per unit time per unit volume is given by,

$$W(t) = \hbar\omega_0\left|\left(\frac{dN_{21}}{dt}\right)_{\text{spontaneous emission}}\right| = A_{21}N_2\hbar\omega_0$$

$$= A_{21}N_{20}e^{-A_{21}t}\hbar\omega_0 = A_{21}N_{20}\hbar\omega_0 e^{-A_{21}t} \quad (17)$$

This equation gives the variation of intensity of the spontaneously emitted radiation. So, the electric field associated with the spontaneous emission can be assumed to be of the form,

$$E(t) = E_0 e^{-\frac{t}{2t_{sp}}} e^{i\omega_0 t} \qquad (18)$$

where, t_{sp} spontaneous emission lifetime and ω_0 is the emission frequency. Let ω be the frequency of the incident radiation. Then the frequency spectrum corresponding to the field given by eqn.18 is obtained by the Fourier transform of the form

$$E(\omega) = \int_0^\infty E_0 e^{-\frac{t}{2t_{sp}}} e^{i\omega_0 t} e^{-i\omega t} dt = \int_0^\infty E_0 e^{\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}t} dt$$

Put, $\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}t = x$

$$dt = \frac{dx}{\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}}$$

Then

$$E(\omega) = \frac{E_0}{i(\omega_0-\omega)-\frac{1}{2t_{sp}}}\int_0^\infty e^x dx = \frac{E_0}{\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}}\left[e^{\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}t}\right]_0^\infty$$

$$= \frac{E_0}{\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}}\left[e^{-\left\{i(\omega-\omega_0)+\frac{1}{2t_{sp}}\right\}t}\right]_0^\infty$$

$$= \frac{E_0}{\left\{i(\omega_0-\omega)-\frac{1}{2t_{sp}}\right\}}[0-1] = \frac{E_0}{\left\{i(\omega-\omega_0)+\frac{1}{2t_{sp}}\right\}} \qquad (19)$$

Thus, the frequency distribution of intensity (power spectrum) is given by,

$$I(\omega) \propto |E(\omega)|^2 = E(\omega)E^*(\omega)$$

$$\propto \left(\frac{E_0}{i(\omega-\omega_0)+\frac{1}{2t_{sp}}}\right)\left(\frac{E_0}{-i(\omega-\omega_0)+\frac{1}{2t_{sp}}}\right)$$

$$\propto \frac{E_0^2}{(\omega-\omega_0)^2+\left(\frac{1}{2t_{sp}}\right)^2}$$

$$= \frac{K}{(\omega-\omega_0)^2+\frac{1}{4t_{sp}^2}} = \frac{4t_{sp}^2 K}{1+4t_{sp}^2(\omega-\omega_0)^2} \qquad (20)$$

By applying the normalization condition, (refer eqn.13 sec.1.2.1)

$$\int g(\omega)d\omega = 1$$

the proportionality constant K can be evaluated as,

$$K = \frac{1}{2\pi t_{sp}} \quad (21)$$

Then the normalized lineshape function is

$$g(\omega) = \frac{2t_{sp}}{\pi}\left[\frac{1}{1+4(\omega-\omega_0)^2 t_{sp}^2}\right] \quad (22)$$

This is again Lorentzian. Using $\frac{dg}{d\omega}=0$, we get the peak of g(ω) lies at $\omega=\omega_0$ and is equal to $\frac{2t_{sp}}{\pi}$. Then the frequency ω' corresponds to the half maximum is obtained by,

$$\frac{t_{sp}}{\pi} = \frac{2t_{sp}}{\pi}\left[\frac{1}{1+4(\omega'-\omega_0)^2 t_{sp}^2}\right]$$

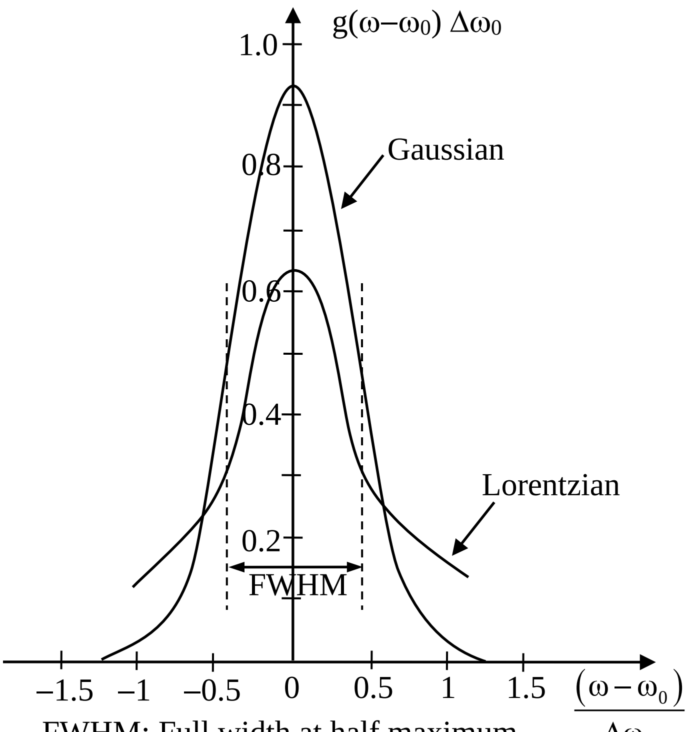
Then, $\omega'-\omega_0 = \Delta\omega = \frac{1}{2t_{sp}}$ (23)

Thus, full width at half maximum (FWHM) is given by,

$$\Delta\omega_N = \frac{1}{t_{sp}} \quad (24)$$

Then eqn.22 becomes,

$$g(\omega) = \frac{2}{\pi\Delta\omega_N}\left[\frac{1}{1+4\left(\frac{\omega-\omega_0}{\Delta\omega_N}\right)^2}\right] \quad (25)$$



FWHM: Full width at half maximum

4. Voigt profile: In general, all the three mechanisms may be present simultaneously and the resultant line-shape function can be obtained by performing a convolution of the different line shapes, known as *voigt profile*.

If any one of the broadening mechanisms dominates over the others, then the line-shape function would correspond to the dominant mechanism. For example, the Doppler-broadened line width corresponding to the 6328Å transition of Ne in He-Ne laser can be calculated using eqn.14. It is about 1700 MHz. For this transition the line width due to collision broadening at a pressure of 0.5 Torr is about 0.64 MHz, whereas that for natural broadening is about 20 MHz. Thus for He-Ne laser the Doppler broadening dominates over natural and collision broadening.

The line broadening mechanisms we have seen is again broadly classified under homogeneous and inhomogeneous broadening. In the case of homogeneous broadening the response of each atom is identical. Certain line broadening mechanisms like collision broadening or natural broadening come under the class of homogeneous broadening, which have

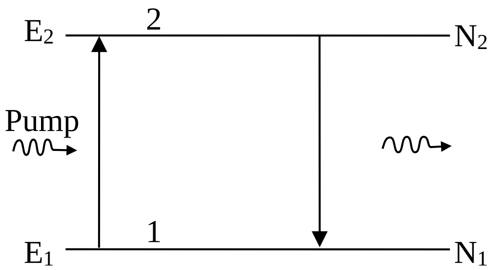
Lorentzian line shape. In the case of inhomogeneous broadening the response of different atoms are different. Doppler broadening or broadening produced by local inhomogeneities in a crystal lattice come under the class of inhomogeneous broadening and their line shapes are Gaussian. [For further study refer chapter-4: Radiative transitions and emission line width: Laser fundamentals- second edition by William T Silfvast].

1.5 Laser rate equations

The laser rate equations govern the rate at which the populations of various levels change under the action of a pump (refer optical pumping) and in presence of the laser radiation. It provides a convenient means of studying the time dependence of the atomic populations of various levels under the presence of radiation at frequencies corresponding to the different transitions of the atoms. It also gives the steady state population difference between the actual levels involved in the laser transition. This helps one to know whether an inversion of population is achievable in a transition. If the population inversion is achievable one can find out the minimum pumping rate required for the continuous wave operation of the laser.

1.5.1 The Two-Level System

We first consider a two-level system consisting of energy levels E_1 and E_2. Let N_1 and N_2, respectively, be the number of atoms per unit volume. Let a monochromatic radiation of frequency ω with energy density u be incident on the system. Then the number of induced absorptions per unit volume per unit time is given by eqn.18 sec.1.2.1.



$$\Gamma_{12} = \frac{\pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g(\omega) u N_1 = W_{12} N_1 \tag{1}$$

where,

$$W_{12} = \frac{\pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g(\omega) u \tag{2}$$

The number of stimulated emissions from E_2 to E_1 per unit time per unit volume is given by eqn. 16 sec.1.2.1.

$$\Gamma_{21} = \frac{\pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g(\omega) u N_2 = W_{21} N_2 = W_{12} N_2 \tag{3}$$

since,

$$W_{21} = \frac{\pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g(\omega) u = W_{12} \tag{4}$$

In addition to these two transitions, there is spontaneous transition from E_2 to E_1. This includes the radiative and nonradiative transitions. The number of spontaneous transitions per unit time per unit volume is proportional to N_2. That is, by eqn.3 sec.1.2

$$U_{21} = T_{21} N_2 \tag{5}$$

Since there are radiative and nonradiative transitions,

$$T_{21} = A_{21} + S_{21} \tag{6}$$

Thus, we can write the rate of change populations in the two energy levels as,

$$\frac{dN_2}{dt} = \Gamma_{12} - \Gamma_{21} - U_{21} = W_{12}(N_1 - N_2) - T_{21} N_2 \tag{7}$$

and, $$\frac{dN_1}{dt} = -\Gamma_{12} + \Gamma_{21} + U_{21} = -W_{12}\left(N_1 - N_2\right) + T_{21}N_2 \quad (8)$$

Adding eqns.7 and 8, we get,

$$\frac{d}{dt}\left(N_1 + N_2\right) = 0$$

i.e. $$N_1 + N_2 = \text{constant} = N \quad (9)$$

Eqn.9 is nothing but it is the fact that the total number of atoms per unit volume is a constant.

Steady state case: At steady state,

$$\frac{dN_1}{dt} = 0 = \frac{dN_2}{dt}$$

Then from eqn.7 we get,

$$W_{12}\left(N_1 - N_2\right) - T_{21}N_2 = 0$$

i.e. $$W_{12}N_1 - \left(W_{12} + T_{21}\right)N_2 = 0$$

Or, $$\frac{N_2}{N_1} = \frac{W_{12}}{W_{12} + T_{21}} \quad (10)$$

Since W_{12} and T_{21} are positive quantities, N_2 is always less than N_1. That is, we can never achieve a steady state population inversion by optical pumping between just two levels. For steady state populations, by subtracting 1 from eqn.10, we can write,

$$\frac{N_2}{N_1} - 1 = \frac{W_{12}}{W_{12} + T_{21}} - 1$$

i.e. $$\frac{N_2 - N_1}{N_1} = \frac{W_{12} - W_{12} - T_{21}}{W_{12} + T_{21}} = -\frac{T_{21}}{W_{12} + T_{21}} \quad (11)$$

Similarly adding 1 to eqn.10, we get,

$$\frac{N_2 + N_1}{N_1} = \frac{W_{12} + W_{12} + T_{21}}{W_{12} + T_{21}} = \frac{2W_{12} + T_{21}}{W_{12} + T_{21}} \quad (12)$$

Dividing eqn.11 by eqn.12 we get,

$$\frac{N_2 - N_1}{N_2 + N_1} = -\frac{T_{21}}{2W_{12} + T_{21}}$$

i.e. $$\frac{\Delta N}{N} = -\frac{1}{1 + \frac{2W_{12}}{T_{21}}} \quad (13)$$

Here, $\Delta N = N_2 - N_1$ is the population difference between the two levels. If we assume that the transitions from the levels E_2 to E_1 is mostly radiative (spontaneous radiative), by eqn.6,

$$T_{21} = A_{21} + S_{21} \approx A_{21} \quad (14)$$

Now we introduce a lineshape function, $\tilde{g}\left(\omega\right)$ such that it is normalized to have a value equal to 1 at the center of the line where $\omega = \omega_0$. That is,

$$\tilde{g}(\omega) = \frac{g(\omega)}{g(\omega_0)} \tag{15}$$

Since $g(\omega) \le g(\omega_0)$ for all ω, we have $0 < \tilde{g}(\omega) < 1$. Using eqn.27 sec.1.3 and eqn.15 in eqn.2 we get,

$$W_{12} = \frac{\pi^2 c^3}{\hbar\omega^3\mu_0^3 t_{sp}} g(\omega) u = \frac{\pi^2 c^3}{\hbar\omega^3\mu_0^3 t_{sp}} \tilde{g}(\omega) g(\omega_0) \frac{\mu_0 I}{c}$$

$$\frac{W_{12}}{T_{21}} = \frac{\pi^2 c^3}{\hbar\omega^3\mu_0^3 t_{sp}} \tilde{g}(\omega) g(\omega_0) \frac{\mu_0 I}{cT_{21}}$$

Using eqn.14 and eqn.12a sec.1.2,

$$= \frac{\pi^2 c^3}{\hbar\omega^3\mu_0^3 t_{sp}} \tilde{g}(\omega) g(\omega_0) \frac{\mu_0 I}{cA_{21}} = \frac{\pi^2 c^2}{\hbar\omega^3\mu_0^2} \tilde{g}(\omega) g(\omega_0) I \tag{16}$$

Using.16 in eqn.13,

$$\frac{\Delta N}{N} = -\frac{1}{1+\frac{2W_{12}}{T_{21}}} = -\frac{1}{1+2\frac{\pi^2 c^2}{\hbar\omega^3\mu_0^2}\tilde{g}(\omega) g(\omega_0) I}$$

$$= -\frac{1}{1+\frac{I}{\frac{\hbar\omega^3\mu_0^2}{2\pi^2 c^2 g(\omega_0)}}\tilde{g}(\omega)} = -\frac{1}{1+\left(\frac{I}{I_s}\right)\tilde{g}(\omega)} \tag{17}$$

where, $$I_s = \frac{\hbar\omega^3\mu_0^2}{2\pi^2 c^2 g(\omega_0)} \tag{18}$$

is called the saturation intensity. In order to understand what I_s represents we consider the case of the interaction of a monochromatic wave of frequency ω_0 with a two-level system. In this case by eqn.15, $\tilde{g}(\omega) = 1$. Then eqn.17 becomes,

$$\frac{\Delta N}{N} = -\frac{1}{1+\left(\frac{I}{I_s}\right)} \tag{19}$$

Now we consider the following three cases.

Case-1: If I << I_s, then the difference in the population densities of the two levels ΔN is independent of the intensity of the incident radiation.

Case-2: If I is comparable to I_s, ΔN becomes a function of I.

Case-3: If $I = I_s$, ΔN has a value half that of the value for very low incident intensities (case-1).

We have, by eqn.29 sec.1.3, the loss/gain coefficient α in terms of ΔN as,

$$\alpha = -\frac{\pi^2 c^2}{\omega^2\mu_0^2 t_{sp}} g(\omega)\Delta N$$

Using eqn.17,

$$= \frac{\pi^2 c^2}{\omega^2\mu_0^2 t_{sp}} \frac{g(\omega) N}{1+\left(\frac{I}{I_s}\right)\tilde{g}(\omega)} = \frac{\alpha_0}{1+\left(\frac{I}{I_s}\right)\tilde{g}(\omega)} \tag{20}$$

where, $$\alpha_0 = \frac{\pi^2 c^2}{\omega^2\mu_0^2 t_{sp}} g(\omega) N \tag{21}$$

For small incident intensities, i.e. $I << I_s$, by eqn.20, α reduces to α_0. Or, α_0 is the loss coefficient corresponds to small signal loss.
By eqn.28 sec.1.3, we have,

$$\frac{dI}{dz} = -\alpha I = -\frac{\alpha_0 I}{1+\left(\frac{I}{I_s}\right)\tilde{g}(\omega)} \tag{22}$$

For $I << I_s$, we get the variation of I with z is exponential and for $I = I_s$, the variation is linear. Thus, we see that the attenuation caused by a medium decreases as the incident intensity increases to a value comparable to the saturation intensity. Later we will see that organic dyes having reasonably low values of I_s are used as saturable absorbers in mode locking and Q-switching of lasers.

Problem: Obtain the variation of I with z.

We can write eqn.22 as,

$$\left\{1+\left(\frac{I}{I_s}\right)\tilde{g}(\omega)\right\}\frac{dI}{I} = -\alpha_0 dz$$

i.e.
$$\frac{dI}{I}+\frac{\tilde{g}(\omega)}{I_s}dI = -\alpha_0 dz$$

Integrating we get,

$$\int\frac{dI}{I}+\frac{\tilde{g}(\omega)}{I_s}\int dI = -\alpha_0\int dz$$

i.e.
$$\ln(I)+\frac{\tilde{g}(\omega)}{I_s}I = -\alpha_0 z + C \tag{23}$$

The constant of integration C can be evaluated by applying the initial condition that $I = I_0$ when $z = 0$. That is,

$$\ln(I_0)+\frac{\tilde{g}(\omega)}{I_s}I_0 = C$$

Then eqn.23 can be written as,

$$\ln(I)+\frac{\tilde{g}(\omega)}{I_s}I = -\alpha_0 z+\ln(I_0)+\frac{\tilde{g}(\omega)}{I_s}I_0$$

i.e.
$$\ln\left(\frac{I}{I_0}\right)+\frac{\tilde{g}(\omega)}{I_s}(I-I_0) = -\alpha_0 z \tag{24}$$

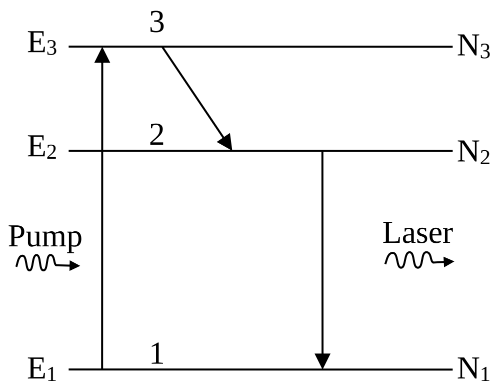
Eqn.24 gives the variation of I with z. The first term in the LHS corresponds to the exponential variation and the second term that of the linear variation.

1.5.2 Three-Level Laser System

Consider a three-level laser system as shown in the figure. All the three levels are assumed as nondegenerate. (Each energy eigen value has only one wave function. Not more than one eigen functions have same energy eigen value). the optical pumping is applied for the $1 \rightarrow 3$ transition. The lasing transition is $2 \rightarrow 1$. The pump lifts the atoms from the level 1 to the level 3. Atoms in the level 3 undergo nonradiative transition to the level 2 rapidly. For lasing transition to take place the level 2 should be metastable and there are more atoms in the metastable level 2 than the atoms in the ground level 1. The level 3 may be a broad level or a group of levels.

Let N_1, N_2 and N_3 be the number of atoms per unit volume in the levels with energy values E_1, E_2 and E_3 respectively. Since only these three levels are populated and the transitions take place only between these levels, we can write,

$$N_1 + N_2 + N_3 = N \qquad (1)$$



where, N is the total number of atoms per unit volume. Number of atoms in the level 3 may be changed because of the transition 1→3 by induced absorption, stimulated emission 3→1 and the nonradiative transition 3→2. Thus,

Rate of change of population of level 3,

$$\frac{dN_3}{dt} = W_p N_1 - W_p N_3 - T_{32} N_3$$

$$= W_p (N_1 - N_3) - T_{32} N_3 \qquad (2)$$

where, W_p is a quantity proportional to the Einstein coefficient B_{13} and the pump beam energy density. (Suffix p stands for pumping transition). W_pN_1 represents the number of induced absorptions per unit time per unit volume due to 1→3 transition. W_pN_3 represents the number of stimulated emissions per unit time per unit volume associated with 3→1 transition. Spontaneous transition 3→1 is neglected because in practical laser systems the atoms in the level 3 almost instantaneously undergo nonradiative transition to level 2. The term $T_{32}N_3$ represents the number of atoms that undergo transition 3→2 per unit time per unit volume. Now we write,

$$T_{32} = A_{32} + S_{32} \qquad (3)$$

where, A_{32} is the Einstein coefficient corresponding to the spontaneous transition 3→2 and S_{32} represents the nonradiative transition rate from level 3 to level 2.

Rate of change of population of level 2, $\frac{dN_2}{dt} = W_l N_1 - W_l N_2 + T_{32} N_3 - T_{21} N_2$

$$= W_l (N_1 - N_2) + T_{32} N_3 - T_{21} N_2 \qquad (4)$$

where, $W_l N_1$ gives the number of atoms per unit time per unit volume coming from level 1by stimulated transition. (Suffix *l* stands for laser transition). The term $W_l N_2$ represents number of atoms leaving from level 2 per unit time per unit volume by induced transition 2→1. Here W_l represents the stimulated transition rate per atom between levels 1 and 2. By referring eqn.2 sec.1.5.1 and eqn.27 sec.1.3 we can write,

$$W_l = \frac{\pi^2 c^3}{\hbar\omega^3 \mu_0^3 t_{sp}} g_1(\omega) \frac{\mu_0 I_l}{c} = \frac{\pi^2 c^2}{\hbar\omega^3 \mu_0^2} A_{21} g_1(\omega) I_l \qquad (5)$$

where, I_l is the intensity of radiation in the 2→1 transition and $g_1(\omega)$ is the lineshape function describing the transitions between levels 1 and 2. Further,

$$T_{21} = A_{21} + S_{21} \qquad (6)$$

The term $T_{32}N_3$ represents the number of atoms that undergo transition 3→2 per unit time per unit volume and $T_{21}N_2$ gives the number of atoms undergo spontaneous transition from 2→1 per unit time per unit volume. The quantity W_l is proportional to Einstein coefficient B_{21} (refer eqn.12 sec.1.2) and the energy density associated with the lasing transition $2\rightarrow1$. The quantity T_{21} represents the net spontaneous relaxation rate from level 2 to level 1. If this transition is predominantly radiative T_{21} is approximately the same as the Einstein coefficient A_{21}.

Similarly,

$$\text{Rate of change of population of level 1, } \frac{dN_1}{dt} = W_p N_3 - W_p N_1 + W_l N_2 - W_l N_1 + T_{21} N_2$$

$$= W_p (N_3 - N_1) + W_l (N_2 - N_1) + T_{21} N_2 \qquad (7)$$

where, the first term represents the stimulated transitions between levels 1 and 3 by optical pumping, the second term represents the stimulated transition by lasing action and the third term represents spontaneous transition from level 2 to level 1.

By adding eqns.2, 4 and 7 we can easily show that,

$$\frac{dN_1}{dt} + \frac{dN_2}{dt} + \frac{dN_3}{dt} = 0 \qquad (8)$$

Eqn.8 is consistent with eqn.1. Eqns.2, 4 and 7 are referred to as the *rate equations*. These equations give the rate change of populations of the three levels in a three-level laser system in terms of W_p and W_l.

To solve these for N_1, N_2 and N_3 we use the steady state conditions. At the steady state $\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = 0$. Then from eqn.2 we get,

$$W_p N_1 - W_p N_3 - T_{32} N_3 = 0$$

i.e.

$$(W_p + T_{32}) N_3 = W_p N_1$$

$$N_3 = \left(\frac{W_p}{W_p + T_{32}} \right) N_1 \qquad (9)$$

From eqn.4 we get,

$$W_l N_1 - W_l N_2 + T_{32} N_3 - T_{21} N_2 = 0$$

i.e.

$$(W_l + T_{21}) N_2 = W_l N_1 + T_{32} N_3 = W_l N_1 + T_{32} \left(\frac{W_p}{W_p + T_{32}} \right) N_1$$

$$= \left[W_l + \left(\frac{T_{32} W_p}{W_p + T_{32}} \right) \right] N_1$$

i.e.

$$N_2 = \left[W_l + \left(\frac{T_{32} W_p}{W_p + T_{32}} \right) \right] \frac{N_1}{(W_l + T_{21})} \qquad (10)$$

$$= \left[\frac{W_l}{(W_l + T_{21})} + \frac{T_{32} W_p}{(W_p + T_{32})(W_l + T_{21})} \right] N_1$$

$$N_2 - N_1 = \left[\frac{W_l}{(W_l + T_{21})} + \frac{T_{32} W_p}{(W_p + T_{32})(W_l + T_{21})} \right] N_1 - N_1$$

$$= \left[\frac{W_l}{(W_l + T_{21})} + \frac{T_{32} W_p}{(W_p + T_{32})(W_l + T_{21})} - 1 \right] N_1$$

$$= \left[\frac{W_l\left(W_p+T_{32}\right)+T_{32}W_p-\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)}{\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)}\right]N_1$$

$$N = N_1 + N_2 + N_3 = \left[\frac{W_l\left(W_p+T_{32}\right)+T_{32}W_p+\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)}{\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)}\right]N_1+\left(\frac{W_p}{W_p+T_{32}}\right)N_1$$

$$= \left[\frac{W_l\left(W_p+T_{32}\right)+T_{32}W_p+\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)+W_p\left(W_l+T_{21}\right)}{\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)}\right]N_1$$

$$\frac{N_2-N_1}{N} = \frac{W_l\left(W_p+T_{32}\right)+T_{32}W_p-\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)}{W_l\left(W_p+T_{32}\right)+T_{32}W_p+\left(W_p+T_{32}\right)\left(W_l+T_{21}\right)+W_p\left(W_l+T_{21}\right)}$$

$$= \frac{W_p\left(T_{32}-T_{21}\right)-T_{32}T_{21}}{3W_pW_l+2W_pT_{21}+W_pT_{32}+2W_lT_{32}+T_{21}T_{32}} \qquad (11)$$

Necessary conditions for population inversion: In order to obtain population inversion $N_2 > N_1$. That is $N_2 - N_1$ must be positive. From eqn.11 it is clear that, this is possible only if the following two conditions are satisfied.

Condition.1: $T_{32} > T_{21}$ (12a)

Since the relaxation times of atoms in levels 3 and 2 are inversely proportional to the corresponding relaxation rates, according to eqn.12a, in order to attain population inversion the lifetime of level 3 must be at least smaller than the lifetime of level 2.

Condition.2: $W_p\left(T_{32}-T_{21}\right) > T_{32}T_{21}$ (12b)

In order to attain population inversion a minimum pump power is required. According to eqn.12b, the minimum pump power required is given by,

$$W_{pt}\left(T_{32}-T_{21}\right) = T_{32}T_{21}$$

Minimum pump power, $W_{pt} = \dfrac{T_{32}T_{21}}{T_{32}-T_{21}}$ (13a)

If $T_{32} >> T_{21}$, $W_{pt} \approx T_{21}$ (13b)

That is, for obtaining population inversion W_p should be greater than W_{pt}. [Suffix t for threshold].

Under the same approximation, eqn.11 becomes,

$$\frac{N_2-N_1}{N} \approx \frac{W_pT_{32}-T_{32}T_{21}}{3W_pW_l+W_pT_{32}+2W_lT_{32}+T_{21}T_{32}}$$

$$\approx \frac{\dfrac{\left(W_p-T_{21}\right)T_{32}}{\left(W_p+T_{21}\right)T_{32}}}{\dfrac{3W_pW_l+\left(W_p+T_{21}\right)T_{32}+2W_lT_{32}}{\left(W_p+T_{21}\right)T_{32}}} = \frac{\dfrac{\left(W_p-T_{21}\right)}{\left(W_p+T_{21}\right)}}{1+\dfrac{\left(3W_p+2T_{32}\right)W_l}{\left(W_p+T_{21}\right)T_{32}}} \qquad (14a)$$

Special cases: *For low laser powers*, i.e. when W_l very small compared to T_{21} (rate of loss of energy by spontaneous transition from level 2 to level 1), we can neglect terms containing W_l in eqn.11. Thus,

$$\frac{N_2 - N_1}{N} \approx \frac{W_p\left(T_{32} - T_{21}\right) - T_{32}T_{21}}{2W_pT_{21} + W_pT_{32} + T_{21}T_{32}}$$

We also assume that $T_{32} >> T_{21}$. Then,

$$\frac{N_2 - N_1}{N} \approx \frac{W_pT_{32} - T_{32}T_{21}}{W_pT_{32} + T_{21}T_{32}} = \frac{W_p - T_{21}}{W_p + T_{21}} \qquad (14b)$$

(We may get this result from eqn.14a by simply $W_l \to 0$).

Under this approximation, for population inversion, we must have $W_p > T_{21}$. From eqn.14b it is clear that the population inversion is independent of the energy corresponding to the laser transition (W_l). Since $N_1 - N_2$ is negative α_ω is negative (eqn.29 sec.1.3) and hence by eqn.30, sec.1.3, the intensity of the beam grows exponentially.

By eqn.29 sec1.3

$$\gamma = -\alpha_\omega = \frac{\pi^2c^2}{\omega^2\mu_0^2t_{sp}}g\left(\omega\right)\left(N_2 - N_1\right)$$

Using eqn.14a,

$$\gamma = \frac{\pi^2c^2}{\omega^2\mu_0^2t_{sp}}g\left(\omega\right)N\frac{\dfrac{\left(W_p - T_{21}\right)}{\left(W_p + T_{21}\right)}}{1+\dfrac{\left(3W_p+2T_{32}\right)W_l}{\left(W_p+T_{21}\right)T_{32}}} = \frac{\gamma_0}{1+\dfrac{\left(3W_p+2T_{32}\right)W_l}{\left(W_p+T_{21}\right)T_{32}}} \qquad (15)$$

where,

$$\gamma_0 = \frac{\pi^2c^2}{\omega^2\mu_0^2t_{sp}}g\left(\omega\right)N\frac{\left(W_p - T_{21}\right)}{\left(W_p + T_{21}\right)} \qquad (16)$$

Using eqn.5 in eqn.15, with $g_1\left(\omega\right) = g\left(\omega\right)$ and $I_l = I$, we get,

$$\gamma = \frac{\gamma_0}{1+\dfrac{\left(3W_p+2T_{32}\right)\dfrac{\pi^2c^2}{\hbar\omega^3\mu_0^2}A_{21}g\left(\omega\right)I}{\left(W_p+T_{21}\right)T_{32}}}$$

Using eqn.15 sec.1.5.1,

$$= \frac{\gamma_0}{1+\dfrac{\left(3W_p+2T_{32}\right)\dfrac{\pi^2c^2}{\hbar\omega^3\mu_0^2}A_{21}\tilde{g}\left(\omega\right)g\left(\omega_0\right)I}{\left(W_p+T_{21}\right)T_{32}}} = \frac{\gamma_0}{1+\dfrac{\tilde{g}\left(\omega\right)I}{\dfrac{\hbar\omega^3\mu_0^2\left(W_p+T_{21}\right)T_{32}}{\pi^2c^2A_{21}g\left(\omega_0\right)\left(3W_p+2T_{32}\right)}}}$$

$$= \frac{\gamma_0}{1+\left(\dfrac{I}{I_s}\right)\tilde{g}\left(\omega\right)} \qquad (17)$$

where,

$$I_s = \frac{\hbar\omega^3\mu_0^2\left(W_p+T_{21}\right)T_{32}}{\pi^2c^2A_{21}g\left(\omega_0\right)\left(3W_p+2T_{32}\right)} \qquad (18)$$

is the saturation ntensity.

For high laser powers: Assuming $T_{32} >> T_{21}$, eqn.11 becomes,

$$\frac{N_2 - N_1}{N} \approx \frac{W_pT_{32} - T_{32}T_{21}}{3W_pW_l + W_pT_{32} + 2W_lT_{32}}$$

Assuming, $W_l >> W_p$,

$$\frac{N_2 - N_1}{N} \approx \frac{W_pT_{32} - T_{32}T_{21}}{3W_pW_l + 2W_lT_{32}} = \frac{W_pT_{32} - T_{32}T_{21}}{W_l\left(3W_p + 2T_{32}\right)} \qquad (19)$$

Under this approximation, the population inversion is inversely proportional to W_l. Hence α_ω is negative, but its value is inversely proportional to W_l. In such cases $\frac{dI_\omega}{dz}$ in eqn.26, sec.1.3, is independent of I_ω and the intensity of the beam grows only linearly with distance.

Now we rewrite eqn.14b as,

$$\frac{\Delta N}{N} \approx \frac{W_p - T_{21}}{W_p + T_{21}}$$

Or, $$NW_p - NT_{21} \approx \Delta NW_p + \Delta NT_{21}$$

$$W_p\left(\frac{N - \Delta N}{2}\right) \approx T_{21}\left(\frac{N + \Delta N}{2}\right) \qquad (20)$$

where, $\Delta N = N_2 - N_1$. Cancelling 2 from both sides, eqn.16 becomes,

$$W_p\left(N - \Delta N\right) \approx T_{21}\left(N + \Delta N\right) \qquad (21)$$

The LHS of this equation represents the number of atoms per unit volume in level 1 is lifted to level 2 per unit time and the RHS represents the number of atoms per unit volume in the level 3 that decays to level 1 per unit time.

For a three level laser system, since the transition rate at level 3 is very large, atoms of level 3 drops to level 2 so quickly, the number of atoms in the level 3 is very small. Thus,

$$N \approx N_1 + N_2 = 2N_1 + N_2 - N_1 = 2N_1 + \Delta N$$

Or, $$N_1 = \frac{N - \Delta N}{2} \quad \text{and} \quad N_2 = N - N_1 = \frac{N + \Delta N}{2} \qquad (22)$$

Then eqn.20 becomes,

$$W_pN_1 \approx T_{21}N_2 \qquad (23)$$

The LHS of eqn.23 represents the number of atoms being lifted (by the pump) per unit volume per unit time from level 1 to level 2 via level 3 and the RHS corresponds to the spontaneous emission rate per unit volume from level 2 to level 1. These rates must be equal under steady state conditions for $W_l \approx 0$, i.e. below the threshold.

Now we estimate the threshold pumping power required to start the laser oscillations. In this case the threshold inversion is very small compared to N. That is, $N_2 - N_1 << N$, or $N_2 \approx N_1 \approx \frac{N}{2}$. Then by eqn.23, $W_p \approx T_{21}$. Now the number of atoms being pumped per unit time per unit volume from level 1 to level 3 is W_pN_1. If ν_p represents the average pump frequency corresponding to the excitation from E_1 to E_3, the power required per unit volume is,

$$P = W_pN_1h\nu_p \qquad (24)$$

Thus, the threshold pump power for laser oscillation is given by,

$$P_{th} = T_{21}N_1h\nu_p \qquad (25)$$

Also, assuming the transition from level 2 to level 1 is mainly radiative, by eqn.6,

$$T_{21} \approx A_{21}$$

Hence, $$P_{th} \approx A_{21}\frac{N}{2}h\nu_p = \frac{Nh\nu_p}{2t_{sp}} \quad (26)$$

Example: Ruby laser

Ruby laser is a three level laser system. By eqn.14b for population inversion, we must have,

$$W_p > T_{21} \approx A_{21} = \frac{1}{t_{sp}} = \frac{1}{3\times10^{-3}} = 330 \text{ sec}^{-1}. \quad (20)$$

We now calculate the minimum amount of pump power needed to maintain population inversion. To maintain population inversion the loss by spontaneous emission at the level 2 must be compensated. [In eqn.4 we can neglect first term since ΔN is small and second term since N_3 is small].

Rate at which atoms decay from the upper level (laser level) $= N_2T_{21}$

Energy needed to lift a photon from level 1 to level 2 $= h\nu_p$

where, ν_p is the average pump frequency.

Minimum power needed to maintain N_2 atoms in the level 2, $P = N_2T_{21}\, h\nu_p = \frac{N_2h\nu_p}{t_{sp}}$

Since ΔN is small, by eqn.19, $N_2 \approx \frac{N}{2}$

Therefore, the threshold pumping power per unit volume required to maintain population inversion in a three level laser system is,

$$P_t \approx \frac{Nh\nu_p}{2t_{sp}} = \frac{1.6\times10^{19}\times6.63\times10^{-34}\times6.25\times10^{14}}{2\times3\times10^{-3}}$$

$$\approx 1100 \text{ watt/cm}^3.$$

If we assume that the efficiency of the pumping source is 25% and also that only 25% is absorbed in passage through the ruby rod,

Threshold electrical power needed $= \frac{1100}{0.25\times0.25} \approx 18 \text{ kW/cm}^3$

This is consistent with the threshold power determined experimentally.

Under pulsed operation (laser operates in pulses) if we assume that the pumping pulse is much shorter than the lifetime of level 2, then the atoms excited to the laser level do not appreciably decay during the duration of the pulse. Then the

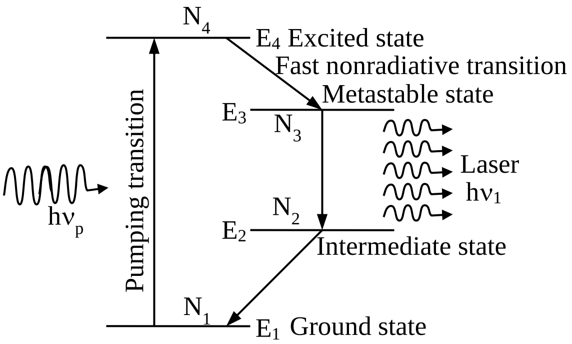
Threshold pump energy, $U_{pt} = \frac{Nh\nu_p}{2}$ per unit volume of the active medium.

For ruby laser, U_{pt} is approximately 54 J/cm^3 (time not 1 sec).

Because of the following factors the ruby laser does not need too large pumping power.

1. The absorption band of ruby crystal is very well matched to the emission spectrum of available pump lamps so that the pumping efficiency is quite high.
2. Most of the atoms pumped to level 3 drop down to level 2, which has a very long lifetime (3×10^{-3} sec), is nearly radiative.
3. Linewidth of the laser transition is also very narrow.

1.5.3 The Four-Level Laser System



The main problem of a three-level laser is that one has to lift more than the 50% of the atoms in the ground level for achieving the population inversion. This is not a problem in a four-level laser. A four-level laser is shown in the figure. Let N_1, N_2, N_3 and N_4 respectively be the populations of the levels with energies E_1, E_2, E_3 and E_4. Level 1 is the ground level and levels 2, 3 and 4 are excited states. Atoms from level 1 are first pumped to level 4 and these atoms then make fast nonradiative relaxations to the level 3, which is the metastable state with longer lifetime. Since the level 2 has a very small lifetime, there is no accumulation of atoms in the level 2 and hence there is a population inversion between the levels 3 and 2. The transition from level 3 to level 2 forms the laser transition.

We now write the rate equations for the different levels. Number of atoms in the level 4 may be changed because of the transition $1\rightarrow4$ by induced absorption, stimulated emission $4\rightarrow1$ and the spontaneous emission and the nonradiative transition from level $4\rightarrow3$. Also, we neglect the spontaneous emissions and nonradiative transitions from level 4 to levels 2 and 1 (i.e. $T_{42} = T_{41} = 0$). Thus, the rate equation for the level 4 is,

$$\frac{dN_4}{dt} = W_p\left(N_1 - N_4\right) - T_{43}N_4 \tag{1}$$

where, $$T_{43} = A_{43} + S_{43} \tag{2}$$

The atoms in the level 3 changes due to radiative (spontaneous emission) and nonradiative transitions from level 4 to level 3, induced emission from level 3 to level 2, induced absorption from level 2 to level 3 and spontaneous emission and nonradiative transition from 3 to 2. Thus, the rate equation for the level 3 can be written as,

$$\frac{dN_3}{dt} = W_l\left(N_2 - N_3\right) + T_{43}N_4 - T_{32}N_3 \tag{3}$$

where, $$W_l = \frac{\pi^2 c^2}{\hbar\omega^3\mu_0^2} A_{32} g_l(\omega) I_l \tag{4}$$

and $$T_{32} = A_{32} + S_{32} \tag{5}$$

The number of atoms in the level 2 changes mainly due to the induced transitions between 3 and 2, the spontaneous and nonradiative transitions from 3 to 2 and spontaneous and nonradiative transition from 2 to 3. Thus,

$$\frac{dN_2}{dt} = -W_l\left(N_2 - N_3\right) + T_{32}N_3 - T_{21}N_2 \tag{6}$$

where, $$T_{21} = A_{21} + S_{21} \tag{7}$$

Finally, in the level 1, we consider only the induced transitions between 4 and 1 due to pumping and the spontaneous and nonradiative transitions between 2 and 1. Thus,

$$\frac{dN_1}{dt} = -W_p\left(N_1 - N_4\right) + T_{21}N_2 \tag{8}$$

Since the total number of toms (in all the levels) is a constant,

$$N = N_1 + N_2 + N_3 + N_4 \tag{9}$$

Under steady state conditions,

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = \frac{dN_4}{dt} = 0 \tag{10}$$

Then by eqn.1,

$$W_p\left(N_1 - N_4\right) = T_{43}N_4$$

i.e. $$W_pN_1 = \left(W_p + T_{43}\right)N_4$$

i.e. $$\frac{N_4}{N_1} = \frac{W_p}{W_p + T_{43}} \tag{11}$$

Or, $$N_4 = \frac{W_p}{W_p + T_{43}}N_1 \tag{12a}$$

If the relaxation from level 4 to level 3 is very rapid,

$$T_{43} >> W_p$$

Then, $$N_4 \approx \frac{W_p}{T_{43}}N_1 \quad ;\text{ i.e. } N_4 << N_1 \text{ ; Or, } N_4 \approx 0 \tag{12b}$$

Applying this approximation to other equations we get,
By eqn.8, using eqn.12,

$$T_{21}N_2 = W_pN_1 - W_p\frac{W_pN_1}{W_p + T_{43}} = \left(\frac{W_pT_{43}}{W_p + T_{43}}\right)N_1$$

i.e. $$N_2 = \frac{1}{T_{21}}\left(\frac{W_pT_{43}}{W_p + T_{43}}\right)N_1 \tag{13a}$$

$$\approx \left(\frac{W_p}{T_{21}}\right)N_1 \tag{13b}$$

By eqn.6, $$N_3 = \frac{\left(W_l + T_{21}\right)}{\left(T_{32} + W_l\right)}N_2 = \frac{\left(W_l + T_{21}\right)}{\left(T_{32} + W_l\right)}\frac{1}{T_{21}}\left(\frac{W_pT_{43}}{W_p + T_{43}}\right)N_1 \tag{14a}$$

$$\approx \frac{\left(W_l + T_{21}\right)}{\left(T_{32} + W_l\right)}\frac{W_p}{T_{21}}N_1 \tag{14b}$$

Then, from eqns.13b and 14b, we get,

$$N_3 - N_2 \approx \frac{\left(W_l + T_{21}\right)}{\left(T_{32} + W_l\right)}\frac{W_p}{T_{21}}N_1 - \left(\frac{W_p}{T_{21}}\right)N_1 = \left\{\frac{\left(T_{21} - T_{32}\right)}{\left(T_{32} + W_l\right)T_{21}}\right\}W_pN_1 \tag{15}$$

By eqns.12b, 13b and 14b

$$N \approx N_1 + N_2 + N_3 + N_4 \approx N_1 + N_2 + N_3$$

$$\approx N_1 + \left(\frac{W_p}{T_{21}}\right)N_1 + \frac{\left(W_l + T_{21}\right)}{\left(T_{32} + W_l\right)}\frac{W_p}{T_{21}}N_1$$

$$\approx \left\{\frac{\left(T_{32} + W_l\right)T_{21} + W_p\left(T_{32} + W_l\right) + \left(W_l + T_{21}\right)W_p}{\left(T_{32} + W_l\right)T_{21}}\right\}N_1$$

$$\approx \left\{\frac{W_p\left(T_{32} + T_{21}\right) + T_{21}T_{32} + W_l\left(T_{21} + 2W_p\right)}{\left(T_{32} + W_l\right)T_{21}}\right\}N_1 \tag{16}$$

Then from eqns.15 and 16,

$$\frac{N_3 - N_2}{N} \approx \frac{W_p\left(T_{21} - T_{32}\right)}{W_p\left(T_{32} + T_{21}\right) + T_{21}T_{32} + W_l\left(T_{21} + 2W_p\right)} \tag{17}$$

Eqn.17 shows that the population inversion between the levels 2 and 3 is possible only if $T_{21} > T_{32}$. If we assume that T_{21} is much greater than T_{32}, we can neglect T_{32} in the sum and the difference terms in eqn.17. Then,

$$\frac{N_3 - N_2}{N} \approx \frac{W_p T_{21}}{W_p T_{21} + T_{21}T_{32} + W_l\left(T_{21} + 2W_p\right)} \approx \frac{W_p}{W_p + T_{32} + \frac{W_l}{T_{21}}\left(T_{21} + 2W_p\right)}$$

$$\approx \left(\frac{W_p}{W_p + T_{32}}\right)\left\{\frac{1}{1 + \frac{W_l\left(T_{21} + 2W_p\right)}{T_{21}\left(W_p + T_{32}\right)}}\right\} \tag{18}$$

From the eqn.18 we see that even for very small pump rates one can obtain population inversion between levels 3 and 2. This is contrary to what we found in the case of a three-level system, where there is a minimum pump rate W_{pt} required to achieve population inversion.

The first factor of eqn.18, which is independent of W_l, gives the small signal gain coefficient, whereas the second factor, which depends on W_l, gives the saturation behavior.

Just below the threshold of the laser oscillation, $W_l = 0$, eqn.18 becomes,

$$\frac{\Delta N}{N} = \frac{N_3 - N_2}{N} \approx \frac{W_p}{W_p + T_{32}} \tag{19}$$

Example.1: The **Nd: YAG laser** is a four-level system with the following parameters. Estimate the threshold pumping rate.

$\lambda_0 = 1.06$ μm; Or, $\nu = 2.83\times10^{14}$ Hz ; $\Delta\nu = 1.95\times10^{11}$ Hz; $\nu_p = 4\times10^{14}$ Hz

$t_{sp} = 2.3\times10^{-4}$ s $N = 6\times10^{19}$ per cm^3; $\mu_0 = 1.82$

Resonator cavity length d = 7 cm; $R_1 = 1$ m; $R_2 = 0.90$ m; Other loss factors neglected.

By eqn.33 sec.1.3.1, since $\alpha_c = 0$,

$$\ln R_1R_2 = 2\alpha_c d - \frac{2\mu_0 d}{ct_c} = -\frac{2\mu_0 d}{ct_c}$$

$$t_c = -\frac{2\mu_0 d}{c\ln R_1R_2} = -\frac{2\times1.82\times7\times10^{-2}}{3\times10^8\ln\left(1\times0.9\right)} = 8.06\times10^{-9}\text{ s}$$

For a Lorentzian line (homogeneous transition)

$$g\left(\omega\right) = \frac{2}{\pi\,\Delta\omega} = \frac{1}{\pi^2\,\Delta\nu} = \frac{1}{\pi^2\times1.95\times10^{11}}$$

By eqn.36b sec1.3.1

i.e. $$N_2 - N_1 \geq \frac{4\mu_0^3\nu^2}{c^3}\left(\frac{t_{sp}}{t_c}\right)\frac{1}{g\left(\omega\right)}$$

i.e. $$\left(\Delta N\right)_{th} = \frac{4\mu_0^3\nu^2}{c^3}\left(\frac{t_{sp}}{t_c}\right)\frac{1}{g\left(\omega\right)}$$

$$= \frac{4\times 1.82^3 \times 2.83^2 \times 10^{28}}{3^3 \times 10^{24}} \left(\frac{2.3\times 10^{-4}}{8.06\times 10^{-9}}\right) \pi^2 \times 1.95\times 10^{11}$$

$$= 3.92\times 10^{21}\ \text{m}^{-3} \qquad = 3.92\times 10^{15}\,\text{cm}^{-3}$$

By eqn.19, $\dfrac{\Delta N}{N} \approx \dfrac{W_p}{W_p + T_{32}}$

Since $(\Delta N)_{th} << N$, $T_{32} >> W_p$, so that we can neglect W_p in the denominator of the above equation. Thus,

$$\frac{(\Delta N)_{th}}{N} \approx \frac{W_{pt}}{T_{32}}$$

i.e. $$W_{pt} \approx \frac{(\Delta N)_{th}}{N} T_{32} = \frac{(\Delta N)_{th}}{N t_{sp}} = \frac{3.92\times 10^{15}\,\text{cm}^{-3}}{6\times 10^{19}\,\text{cm}^{-3} \times 2.3\times 10^{-4}\,\text{s}}$$

$$\approx 0.3\ \text{s}^{-1}$$

At this pumping rate, the number of atoms pumped from 1 to 4 is, $W_{pt}N_1$. Also since N_2, N_3 and N_4 are very small compared with N_1, we can assume $N_1 \approx N$. Then the threshold pump power required per unit volume (cm^{-3}) is,

$$P_{th} = W_{pt} N_1 h\nu_p \approx W_{pt} N h\nu_p$$

$$\approx 0.3\times 6\times 10^{19} \times 6.63\times 10^{-34} \times 4\times 10^{14}$$

$$\approx 4.8\ \text{W/cm}^3$$

This is much less than that for ruby laser.

Example-2: He-Ne laser. Estimate the threshold power required from the following data.

$\lambda_0 = 0.6328$ µm; Or, $\nu = 4.74\times 10^{14}\,\text{Hz}$; $\Delta\nu = 10^9\,\text{Hz}$; $\nu_p = 5\times 10^{15}\,\text{Hz}$

$t_{sp} = 10^{-7}\,\text{s}$; $\mu_0 = 1$

Resonator cavity length d = 10 cm; $R_1 = R_2 = 0.98$ m; Other loss factors neglected.

By eqn.33 sec.1.3.1, since $\alpha_c = 0$,

$$t_c = -\frac{2\mu_0 d}{c \ln R_1 R_2} = -\frac{2\times 1\times 10\times 10^{-2}}{3\times 10^8 \ln(0.98\times 0.98)} = 1.65\times 10^{-8}\,\text{s}$$

For an inhomogeneously broadened transition (for a Gaussian line),

$$g(\omega) = \frac{2(\pi \ln 2)^{1/2}}{\pi\,\Delta\omega} = \frac{(\pi \ln 2)^{1/2}}{\pi^2\ \Delta\nu} = \frac{(3.14\times \ln 2)^{1/2}}{3.14^2 \times 10^9}$$

$$= 1.5\times 10^{-10}\,\text{s}$$

The threshold population inversion required is,

$$(\Delta N)_{th} = \frac{4\mu_0^3 \nu^2}{c^3}\left(\frac{t_{sp}}{t_c}\right)\frac{1}{g(\omega)}$$

$$= \frac{4\times 1^3 \times 4.74^2 \times 10^{28}}{3^3 \times 10^{24}}\left(\frac{10^{-7}}{1.6\times 10^{-8}}\right)\frac{1}{1.5\times 10^{-10}}$$

$$= 1.4\times 10^{15}\,\text{per m}^3 \qquad = 1.4\times 10^9\ \text{per cm}^3$$

$$W_{pt} \approx \frac{(\Delta N)_{th}}{N} T_{32} = \frac{(\Delta N)_{th}}{N t_{sp}}$$

Then the threshold pump power required per unit volume (cm^{-3}) is,

$$P_{th} = W_{pt}N_1h\nu_p \approx W_{pt}Nh\nu_p = \frac{(\Delta N)_{th}}{Nt_{sp}}Nh\nu_p$$

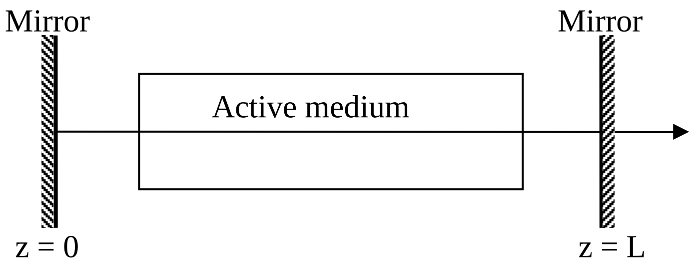
$$= \frac{(\Delta N)_{th}}{t_{sp}}h\nu_p = \frac{1.4\times10^{9}\times6.63\times10^{-34}\times5\times10^{15}}{10^{-7}}$$

$$= 46.41 \text{ mW per cm}^3 \approx 50 \text{ mW per cm}^3.$$

Again, this is much less than that for ruby laser.

1.6 Cavity Modes- Semiclassical theory

Next we deal with the semiclassical theory of the laser developed by Lamb in 1964. In this analysis, we treat the electromagnetic field classically with the help of Maxwell's equations and the atom will be treated using quantum mechanics. We consider a collection of two level atoms placed inside an optical resonator and find out the cavity modes.



Consider an optical resonator consisting of two parallel plane mirrors facing each other as shown in the figure.. The active medium is placed inside the cavity of the resonator. We choose a coordinate system such that its z-axis along the length of the cavity and origin at the centre of one of the mirrors. The plane mirrors are at z = 0 and z = L, where L is the separation between the mirrors.

The Electromagnetic radiation inside the cavity can be described by the Maxwell's equations. In an isotropic, homogeneous medium the equations are,

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad (1)$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \quad (2)$$

$$\nabla.\mathbf{D} = \rho \quad (3)$$

$$\nabla.\mathbf{B} = 0 \quad (4)$$

where, ρ is the free charge density, **J** is the conduction current density **E** is the electric field, **D** is the electric displacement, **B** magnetic induction and **H** is the magnetizing field. Inside the cavity we may assume,

Free charge density, $\rho = 0$ (5)

Magnetic induction, $\mathbf{B} = \mu_0\mathbf{H}$ (6)

Electric displacement vector, $\mathbf{D} = \mathbf{P} + \varepsilon_0\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E}$ (7)

Conduction current density $\mathbf{J} = \sigma\mathbf{E}$ (8)

where, ε_0 is the permittivity of the free space, μ_0 the permeability of free space, **P** the dielectric polarization and σ the conductivity of the medium. The different types of losses like the ohmic loss, loss due to diffraction, loss due to the finite transmission at the mirrors etc. are taken into account in σ. These losses cause the attenuation of the wave. Taking curl of eqn.1,

$$\nabla\times(\nabla\times\mathbf{E}) = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{H})$$

$$= -\mu_0\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J}\right)$$

$$= -\mu_0\frac{\partial^2\mathbf{D}}{\partial t^2} - \mu_0\frac{\partial\mathbf{J}}{\partial t} \tag{9}$$

$$= -\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} - \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}$$

Or,
$$\nabla\times(\nabla\times\mathbf{E})+\mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}+\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2} \tag{10}$$

Step-1: If we assume that the losses in the medium is small we can neglect the term containing σ. Also if we assume that the medium is sufficiently dilute (i.e. the particle separation is large) the field that acts on a particle is the electric field of the wave and the local field created by the polarized surroundings is negligibly small so that we can neglect the term containing **P**. Then eqn.10 becomes,

$$\nabla\times(\nabla\times\mathbf{E})+\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{11}$$

Since **P** is small, eqn.7 becomes, $\mathbf{D} = \varepsilon_0\mathbf{E}$ and in the absence of free charges eqn. 3 gives,

$$\varepsilon_0(\nabla.\mathbf{E}) = 0 \tag{12}$$

Expanding and using eqn.12,

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla.\mathbf{E})-(\nabla.\nabla)\mathbf{E} = -\nabla^2\mathbf{E} \tag{13}$$

Now we assume that the electric field varies in the z-direction only. This is justified because the intensity variations in the directions transverse to the laser axis is small in distances of the order of wavelength λ. Then using eqn.13, eqn.11 becomes,

$$\frac{\partial^2\mathbf{E}}{\partial z^2}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{14}$$

Comparing with the standard wave equation, we get the velocity of the electromagnetic wave in free space as,

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{15}$$

If we assume that the wave is polarized in a specific direction, say $\hat{\mathbf{a}}$. Then $\mathbf{E} = \hat{\mathbf{a}}\,E$ and eqn.14 can be written the scalar form as,

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \tag{16}$$

The eqn.16 is an equation of two variables z and t. To solve eqn.16 by variable separable method, let,

$$E(z,t) = Z(z)T(t) \tag{17}$$

Using eqn.17, eqn.16 becomes,

$$T(t)\frac{d^2Z}{dz^2} = \frac{1}{c^2}Z(z)\frac{d^2T}{dt^2}$$

Dividing throughout by $Z(z)T(t)$, we get,

$$\frac{1}{Z(z)}\frac{d^2Z}{dz^2} = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T}{dt^2}$$

Both sides of the above equation are of different variables, the equation is correct only if both sides are separately equal to the same constant, say, $-k^2$, then

$$\frac{1}{Z(z)}\frac{d^2Z}{dz^2} = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T}{dt^2} = -k^2 \qquad (18)$$

Then we get two separate equations,

$$\frac{d^2Z}{dz^2} + k^2Z = 0 \qquad (18a)$$

And, $$\frac{d^2T}{dt^2} + k^2c^2T(t) = 0 \qquad (18b)$$

where, k is called the wave number (wave vector) defined as, $k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} = \frac{\omega}{c}$ (18c)

The solution of eqn.18a is

$$Z(z) = A\sin(kz+\theta) \qquad (19)$$

Applying the boundary condition that the wave vanishes at the cavity ends (i.e. Z = 0 at z = 0 and z = L).

The condition, Z = 0 at z = 0, gives θ = 0, then

$$Z(z) = A\sin(kz) \qquad (19a)$$

The condition, Z = 0 at z = L, gives, $A\sin(kL) = 0$

i.e. $$kL = n\pi$$

Or, $k_n = \frac{n\pi}{L}$; where, n = 1, 2, 3, ……….. is called the *mode number*. (20)

The solution of eqn.18b is of the form,

$$T(t) = \cos(\Omega_n t)$$

where, $$\Omega_n = k_n c = \frac{n\pi c}{L} \qquad (21)$$

Then the complete solution of eqn.16 is given by,

$$E(z,t) = \sum_n A_n \sin(k_n z)\cos(\Omega_n t) \qquad (22)$$

Step-2: In presence of the different losses the field equation for a dilute medium is given by, (eqn.10 with term containing **P** neglected),

$$\frac{\partial^2E}{\partial z^2} - \mu_0\sigma\frac{\partial E}{\partial t} = \frac{1}{c^2}\frac{\partial^2E}{\partial t^2} \qquad (23)$$

The solution of this equation contains a space dependent part $\sin(k_n z)$ and a time dependent part $e^{i\Gamma_n t}$. Then,

$$E(z,t) = A\sin(k_n z)e^{i\Gamma_n t}$$

$$\frac{\partial E}{\partial t} = i\Gamma_n A\sin(k_n z)e^{i\Gamma_n t}$$

$$\frac{\partial^2 E}{\partial t^2} = -\Gamma_n^2 A\sin(k_n z)e^{i\Gamma_n t}$$

$$\frac{\partial^2 E}{\partial z^2} = -k_n^2 A\sin(k_n z)e^{i\Gamma_n t}$$

Substituting in eqn.23 we get,

$$-k_n^2 A\sin(k_n z)e^{i\Gamma_n t} - i\mu_0\sigma\Gamma_n A\sin(k_n z)e^{i\Gamma_n t} = -\frac{\Gamma_n^2}{c^2}A\sin(k_n z)e^{i\Gamma_n t}$$

i.e. $\Gamma_n^2 - i\mu_0\sigma c^2\Gamma_n - c^2k_n^2 = 0$

Using eqn.21,

$$\Gamma_n^2 - i\mu_0\sigma c^2\Gamma_n - \Omega_n^2 = 0 \qquad (24)$$

The solutions of this quadratic equation are,

$$\Gamma_n = \frac{i\mu_0\sigma c^2 \pm \sqrt{-\mu_0^2\sigma^2c^4 + 4\Omega_n^2}}{2}$$

Using eqn.15, $$= \frac{\frac{i\sigma}{\varepsilon_0} \pm \sqrt{-\frac{\sigma^2}{\varepsilon_0^2} + 4\Omega_n^2}}{2}$$

When σ is small, $$\Gamma_n \approx \frac{i\sigma}{2\varepsilon_0} \pm \Omega_n \qquad (25)$$

Then the time dependent part of the solution is $$e^{i\left(\frac{i\sigma}{2\varepsilon_0} \pm \Omega_n\right)t} = e^{-\frac{\sigma}{2\varepsilon_0}t}e^{\pm i\Omega_n t} = e^{-\frac{\Omega_n}{2Q_n}t}e^{\pm i\Omega_n t} \qquad (26)$$

The real part of this equation is,

$$e^{-\frac{\Omega_n}{2Q_n}t}\cos(\Omega_n t) \qquad (27)$$

where, $$Q_n = \frac{\Omega_n \varepsilon_0}{\sigma} \qquad (28)$$

is the *quality factor*, which will be discussed in detail in the next topic. Thus considering the real part of the eqn.26, the complete solution of eqn.23 is given by,

$$E(z,t) = \sum_n A_n e^{-\frac{\Omega_n}{2Q_n}t}\cos(\Omega_n t)\sin(k_n z) \qquad (29)$$

Step-3: In our final step we find out the solution of eqn.10

$$\frac{\partial^2 E}{\partial z^2} - \mu_0\sigma\frac{\partial E}{\partial t} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0\frac{\partial^2 P}{\partial t^2} \qquad (30a)$$

Multiplying with c^2,

$$c^2\frac{\partial^2 E}{\partial z^2}-\mu_0 c^2\sigma\frac{\partial E}{\partial t}-\frac{\partial^2 E}{\partial t^2} = \mu_0 c^2\frac{\partial^2 P}{\partial t^2} \qquad (30b)$$

As in the cases1 and 2 (eqns.22 and 29) we assume that the solution of this equation is of the form real part of the time dependent part multiplied by $\sin(k_n z)$. That is,

$$E(z,t) = \frac{1}{2}\sum_n\left[\left\{E_n(t)e^{-i\{\omega_n t+\phi_n(t)\}}\right\}+\left\{E_n(t)e^{i\{\omega_n t+\phi_n(t)\}}\right\}\right]\sin(k_n z) \qquad (31)$$

Second term in the RHS is the complex conjugate of the first term. This is to make sure that E is necessarily real. $E_n(t)$ and $\phi_n(t)$ are real slowly varying amplitude and phase coefficients. ω_n is the frequency of oscillation of the mode, which may be, in general, slightly different from Ω_n. Since polarization P is proportional to electric field we assume P to be of the form,

$$P(z,t) = \frac{1}{2}\sum_n\left[\left\{P_n(t,z)e^{-i\{\omega_n t+\phi_n(t)\}}\right\}+\left\{P_n(t,z)e^{i\{\omega_n t+\phi_n(t)\}}\right\}\right] \qquad (32)$$

$P_n(t,z)$ may be complex but is a slowly varying component of the polarization.

$$c^2\frac{\partial^2 E}{\partial z^2} = -\frac{1}{2}c^2k_n^2E_n(t,z) = -\frac{1}{2}\Omega_n^2E_n$$

$$c^2\mu_0\sigma\frac{\partial E}{\partial t} = \frac{\sigma}{\varepsilon_0}\frac{\partial E}{\partial t} \approx -\frac{1}{2}i\left(\frac{\sigma}{\varepsilon_0}\right)\omega_n E_n \text{; All other terms being small neglected.}$$

$$c^2\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial t^2} \approx -i\omega_n\dot{E}_n-\frac{1}{2}\left(\omega_n+\dot{\phi}_n\right)^2E_n \text{; All other terms neglected.}$$

$$c^2\mu_0\frac{\partial^2 P}{\partial t^2} = \frac{1}{\varepsilon_0}\frac{\partial^2 P}{\partial t^2} \approx -\frac{1}{2}\frac{\omega_n^2}{\varepsilon_0}p_n(t)\text{; All other terms neglected.}$$

Substituting in eqn.30b, we get,

$$-\frac{1}{2}\Omega_n^2E_n+\frac{1}{2}i\left(\frac{\sigma}{\varepsilon_0}\right)\omega_nE_n+i\omega_n\dot{E}_n+\frac{1}{2}\left(\omega_n+\dot{\phi}_n\right)^2E_n = -\frac{1}{2}\frac{\omega_n^2}{\varepsilon_0}p_n(t)$$

Multiplying throughout by –2, we get,

$$\Omega_n^2E_n-i\left(\frac{\sigma}{\varepsilon_0}\right)\omega_nE_n-2i\omega_n\dot{E}_n-\left(\omega_n+\dot{\phi}_n\right)^2E_n = \frac{\omega_n^2}{\varepsilon_0}p_n(t) \qquad (33)$$

where, $p_n(t)$ is obtained as the Fourier transform of eqn.32.

i.e. $$p_n(t) = \frac{2}{L}\int_0^L P(z,t)\sin(k_nz)dz$$

Consider, $\Omega_n^2-\left(\omega_n+\dot{\phi}_n\right)^2 = \left\{\Omega_n+\left(\omega_n+\dot{\phi}_n\right)\right\}\left\{\Omega_n-\left(\omega_n+\dot{\phi}_n\right)\right\}$

Since ω_n is very close to Ω_n and $2\omega_n >> \dot{\phi}_n$,

$$\Omega_n^2-\left(\omega_n+\dot{\phi}_n\right)^2 \approx 2\omega_n\left(\Omega_n-\omega_n-\dot{\phi}_n\right) \qquad (34)$$

Then eqn.33 becomes,

$$2\omega_n\left(\Omega_n-\omega_n-\dot{\phi}_n\right)E_n-i\left(\frac{\sigma}{\varepsilon_0}\right)\omega_nE_n-2i\omega_n\dot{E}_n = \frac{\omega_n^2}{\varepsilon_0}p_n(t)$$

Equating the real and imaginary parts of both sides, we get,

$$\left(\omega_n + \dot{\phi}_n - \Omega_n\right) E_n(t) = -\frac{\omega_n}{2\varepsilon_0} \mathrm{Re}\{p_n(t)\} \tag{35}$$

$$\dot{E}_n + \left(\frac{\sigma}{2\varepsilon_0}\right) E_n = -\frac{\omega_n}{2\varepsilon_0} \mathrm{Im}\{p_n(t)\}$$

i.e.
$$\dot{E}_n(t) + \left(\frac{\omega_n}{2Q'_n}\right) E_n(t) = -\frac{\omega_n}{2\varepsilon_0} \mathrm{Im}\{p_n(t)\} \tag{36}$$

where,
$$Q'_n = \frac{\varepsilon_0 \omega_n}{\sigma} \tag{37}$$

When $p_n = 0$, (real part of p_n = Imaginary part of $p_n = 0$), then from eqn.35, we get,

$$\Omega_n = \omega_n + \dot{\phi}_n \approx \omega_n$$

And from eqn.36, we obtain,

$$\frac{dE_n}{dt} = -\left(\frac{\omega_n}{2Q'_n}\right) E_n$$

i.e.
$$\frac{dE_n}{E_n} = -\left(\frac{\omega_n}{2Q'_n}\right) dt$$

Integrating and taking exponential we get,

$$E_n = E_n^0 e^{-\left(\frac{\omega_n}{2Q'_n}\right)t}$$

That is, E_n decreases exponentially with time. These are consistent with our earlier findings (step-2).

Polarization is proportional to the total electric field. That is,

$$p_n(t) = \varepsilon_0 \chi_n E_n(t) = \varepsilon_0 \left(\chi'_n + i\chi''_n\right) E_n(t) \tag{38}$$

Since, in general, the *susceptibility* χ_n is complex. Then from eqn.35, we get,

$$\left(\omega_n + \dot{\phi}_n - \Omega_n\right) E_n(t) = -\frac{\omega_n}{2\varepsilon_0} \mathrm{Re}\{p_n(t)\} = -\frac{\omega_n}{2\varepsilon_0} \varepsilon_0 \chi'_n E_n(t)$$

$$= -\frac{\omega_n}{2} \chi'_n E_n(t)$$

i.e.
$$\omega_n + \dot{\phi}_n = \Omega_n - \frac{1}{2}\omega_n \chi'_n \tag{39}$$

and from eqn.36 we get,

$$\dot{E}_n(t) + \left(\frac{\omega_n}{2Q'_n}\right) E_n(t) = -\frac{\omega_n}{2\varepsilon_0} \mathrm{Im}\{p_n(t)\} = -\frac{\omega_n}{2\varepsilon_0} \varepsilon_0 \chi''_n E_n(t)$$

i.e.
$$\dot{E}_n(t) = -\frac{1}{2}\left(\frac{\omega_n}{Q'_n}\right) E_n(t) - \frac{1}{2}\omega_n \chi''_n E_n(t) \tag{40}$$

The first term on the RHS represents the cavity losses and the second term gives the effect of the medium filling the cavity. We can see that if χ''_n is positive the cavity medium adds to the losses. On the other hand if χ''_n is negative, the cavity medium reduces the losses.

If
$$\chi''_n = -\frac{1}{Q'_n} \tag{41}$$

The cavity losses are just compensated by the gain of the cavity medium. Thus eqn.41 is referred to as the *threshold condition*. If $-\chi_n'' > \frac{1}{Q_n'}$ there will be buildup of oscillation.

If we neglect the term $\dot{\phi}_n$ in eqn.39, we can see that the oscillation frequency of the active medium differs from the passive cavity frequency Ω_n by $-\frac{1}{2}\omega_n\chi_n'$, which is known as the *pulling term*. In order to understand physically the gain and the pulling effects due to the cavity medium, we consider a plane wave propagating through the cavity medium. The permittivity and the susceptibility of the medium of the cavity are related by the equation,

$$\varepsilon = \varepsilon_0(1+\chi_n) = \varepsilon_r\varepsilon_0 = K\varepsilon_0 \qquad (42)$$

where, K is the dielectric constant of the medium.

Refractive index of the medium, $n = \sqrt{K} = (1+\chi_n)^{1/2} \approx 1+\frac{1}{2}\chi_n = 1+\frac{1}{2}\chi_n' + i\frac{1}{2}\chi_n'' \qquad (43)$

Then the propagation constant of the plane wave in such a medium is given by,

$$k = \frac{2\pi}{\lambda} = \frac{n\omega}{c} = \frac{\omega}{c}\sqrt{K} \approx \frac{\omega}{c}\left(1+\frac{1}{2}\chi_n' + i\frac{1}{2}\chi_n''\right)$$

$$= \frac{\omega}{c}\left(1+\frac{1}{2}\chi_n'\right) + i\frac{1}{2}\frac{\omega}{c}\chi_n'' = \alpha + i\delta \qquad (44)$$

Where, $\alpha = \frac{\omega}{c}\left(1+\frac{1}{2}\chi_n'\right)$ and $\delta = \frac{\omega}{2c}\chi_n''$ (45)

Thus, a plane wave propagating along the z-direction would have the z dependence of the form,

$$e^{ikz} = e^{i(\alpha+i\delta)z} = e^{i\alpha z}e^{-\delta z} \qquad (46)$$

In the absence of the component due to laser transition, $\chi_n' = \chi_n'' = 0$, then $\alpha = \frac{\omega}{c}$ and $\delta = 0$. Then the plane wave propagating through the medium undergoes a phase shift per unit length of $\frac{\omega}{c}$. The presence of the laser transition contributes both to the phase change and also to the loss or amplification of the beam. Thus if χ_n'' is positive, δ is positive and the beam gets attenuated (eqn.46) as it propagates along the z-direction. On the other hand if χ_n'' is negative, δ is negative and the beam gets amplified as it propagates through the medium. As the response of the medium is stimulated by the field, the applied field and the stimulated response are phase coherent.

In addition to the losses or amplification, mentioned above, caused by the cavity medium there is also a phase shift due to the real part of the susceptibility χ_n'. That is, the frequency of the oscillating mode is at the centre of the atomic line and it has opposite signs on either sides of the line centre. This additional phase shift causes the frequencies of oscillation of the optical cavity filled with the laser medium to be different from the frequencies of oscillation of the cavity in the absence of the laser medium. Thus the actual oscillation frequencies are slightly pulled towards the centre of the atomic line and hence this phenomenon is referred to as *mode pulling*. We can show that at resonance χ_n' is exactly zero.

1.7 Q of cavity

We have already seen that the laser system is formed by a resonator cavity filled with an active medium. Usually the mirrors are used for producing the cavity. Because of several reasons the loss of energy is associated with any mode in such a cavity and the cavity acts as an *open resonator*. The main loss mechanisms are due to,

1. Finite reflectivity of the mirrors. The total energy is partially reflected and partially transmitted.
2. Scattering and absorption in the medium filling the resonator cavity.
3. The diffraction spill-over when the field undergoes reflection from the mirrors.

This dissipation of energy is described in terms of the quality factor Q of the mode, which is defined as,

$$\text{Quality factor,} \quad Q = \omega_0 \times \frac{\text{Energy stored in the mode}}{\text{Energy dissipated per second in that mode}} \qquad (1)$$

where, ω_0 (= $2\pi\nu_0$) corresponds to the oscillation frequency of the mode. Let W(t) be the energy stored in the mode at time t. Then, from eqn.1, we get,

$$\text{Energy dissipated per second in a mode,} \quad \frac{dW}{dt} = -\frac{\omega_0}{Q} W \qquad (2)$$

The negative sign indicates the energy loss. Eqn.2 can be written as,

$$\frac{dW}{W} = -\frac{\omega_0}{Q} dt$$

$$\text{Integrating,} \quad \ln W = -\frac{\omega_0}{Q} t + \ln W_0$$

where, $\ln W_0$ is the constant of integration and $W_0 = W(t = 0)$. Rearranging and taking the exponential of the above equation we get,

$$W(t) = W_0 e^{-\frac{\omega_0}{Q} t} \qquad (3)$$

This corresponds to an exponential decay of energy. Now we define the *passive cavity lifetime* as the time for which the energy of the passive cavity decays to 1/e of its value at t = 0. From eqn.3 we get,

$$\text{Passive cavity lifetime, } t_c = \frac{Q}{\omega_0} = \frac{Q}{2\pi\nu_0} \qquad (4)$$

Considering this exponential loss of energy, the electric field can be expressed as,

$$E(t) = E_0 e^{-\left(\frac{\omega_0}{2Q}\right)t} e^{i\omega_0 t} = E_0 e^{-\left(\frac{\omega_0}{2Q}\right)t + i\omega_0 t} \qquad (5)$$

The frequency spectrum associated with this wave train which extends from t = 0 to t = ∞, is obtained by a Fourier transform of eqn.5. That is,

$$\tilde{E}(\omega) = \int_0^\infty E(t) e^{-i\omega t} dt = \int_0^\infty E_0 e^{-\left(\frac{\omega_0}{2Q}\right)t + i\omega_0 t} e^{-i\omega t} dt$$

$$= \int_0^\infty E_0 e^{-\left\{i(\omega-\omega_0) + \frac{\omega_0}{2Q}\right\}t} dt$$

Integrating we get,

$$\tilde{E}(\omega) = E_0\left[\frac{e^{-\left\{i(\omega-\omega_0)+\frac{\omega_0}{2Q}\right\}t}}{-\left\{i(\omega-\omega_0)+\frac{\omega_0}{2Q}\right\}}\right]_0^\infty = \frac{E_0}{i(\omega-\omega_0)+\frac{\omega_0}{2Q}}$$

Or,
$$\tilde{E}(\nu) = \frac{E_0}{2\pi\left\{i(\nu-\nu_0)+\frac{\nu_0}{2Q}\right\}} \quad (6)$$

And the corresponding frequency distribution of the intensity is,

$$I(\nu) = \left|\tilde{E}(\nu)\right|^2 = \tilde{E}^*(\nu)\tilde{E}(\nu)$$

$$= \left[\frac{E_0}{2\pi\left\{-i(\nu-\nu_0)+\frac{\nu_0}{2Q}\right\}}\right]\left[\frac{E_0}{2\pi\left\{i(\nu-\nu_0)+\frac{\nu_0}{2Q}\right\}}\right]$$

$$= \frac{E_0^2}{4\pi^2\left\{(\nu-\nu_0)^2+\frac{\nu_0^2}{4Q^2}\right\}} \quad (7)$$

As per the eqn.7 the frequency distribution of the intensity is Lorentzian as shown in the figure below. It is peaked at $\nu = \nu_0$. This can be proved by equating the first derivative of $I(\nu)$ with respect to ν to zero. By substituting $\nu = \nu_0$ in eqn.7 we get the peak value.

[
$$I(\nu)_{max} = \frac{E_0^2Q^2}{\pi^2\nu_0^2}$$

Let ν' be the frequency corresponding to the half maximum. Then,

$$\frac{E_0^2Q^2}{2\pi^2\nu_0^2} = \frac{E_0^2}{4\pi^2\left\{(\nu'-\nu_0)^2+\frac{\nu_0^2}{4Q^2}\right\}}$$

Cross multiplying and rearranging we get,

$$\nu'^2 - 2\nu'\nu_0 + \nu_0^2\left(1-\frac{1}{4Q^2}\right) = 0$$

The roots are given by,

$$\nu' = \frac{2\nu_0 \pm \sqrt{4\nu_0^2 - 4\nu_0^2\left(1-\frac{1}{4Q^2}\right)}}{2} = \nu_0 \pm \sqrt{\nu_0^2 - \nu_0^2 + \frac{\nu_0^2}{4Q^2}}$$

$$= \nu_0 \pm \frac{\nu_0}{2Q} = \nu_0\left(1 \pm \frac{1}{2Q}\right)$$

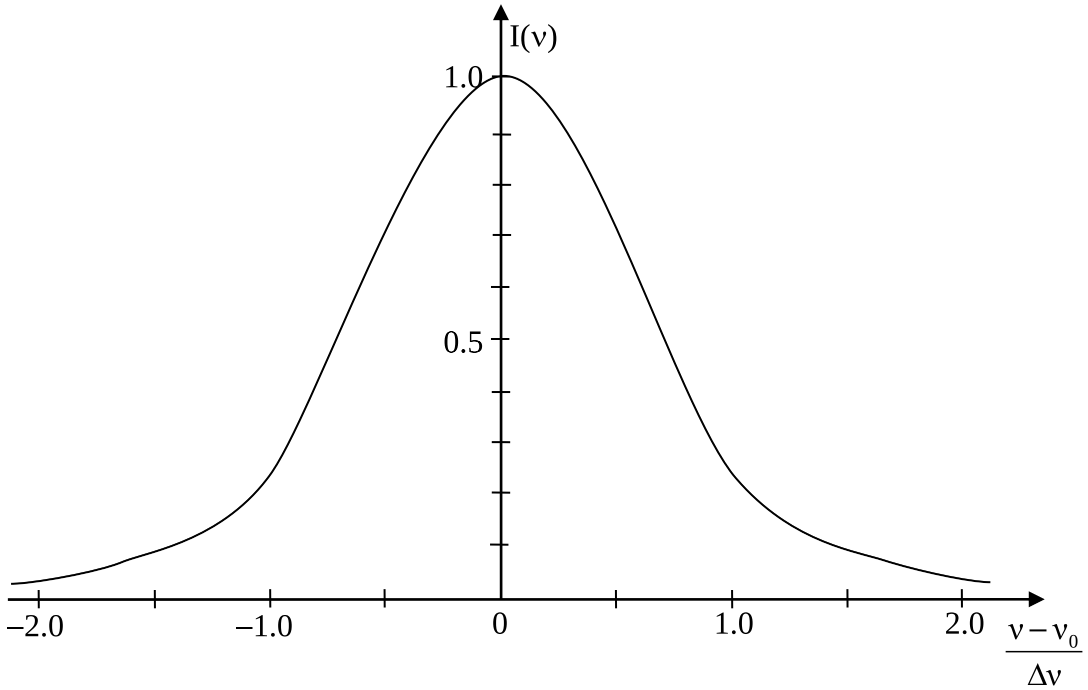
$$\nu_1' = \nu_0\left(1-\frac{1}{2Q}\right) \quad \text{and} \quad \nu_2' = \nu_0\left(1+\frac{1}{2Q}\right)$$

Then, $\Delta\nu = \nu_2' - \nu_1' = \dfrac{\nu_0}{Q}$]

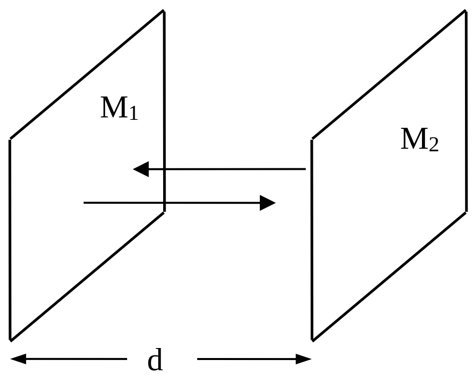
Then we can find out the frequency corresponding to the half maximum, which are, $\nu_0 + \dfrac{\nu_0}{2Q}$ and $\nu_0 - \dfrac{\nu_0}{2Q}$. Thus the full width at the half maximum (FWHM) is,

$$\Delta\nu = \frac{\nu_0}{Q} \tag{8}$$

Eqn.8 shows that the width of the output spectrum depends inversely on the quality factor Q associated with that mode. The smaller the loss in a mode, the higher is the value of Q and hence smaller is the frequency half width.



To calculate Q of a passive resonator: Let W_0 be the total energy contained in the cavity when t = 0. R_1 and R_2 are the power reflection coefficients of the two mirrors M_1 and M_2. Let α_c be the absorption coefficient of the medium per unit length. In one complete cycle there occurs a pair of reflections, (one reflection in each mirror). Let d is the length of the cavity and n_0 be the refractive index of the medium filling the cavity. Then, (refer sec.1.2.3),



Energy remaining in the cavity after one complete cycle, $W(t) = W_0 R_1 R_2 e^{-2\alpha_c d}$ (9)

Velocity of light in the medium, $v = \dfrac{c}{n_0}$

Time taken for one complete cycle (one to and fro travel), $t = \dfrac{2d}{v} = \dfrac{2n_0 d}{c}$ (10)

By eqn.3, the energy in the cavity after one cycle, $W(t) = W_0 e^{-\frac{\omega_0}{Q}t} = W_0 e^{-\frac{2\pi\nu_0}{Q}\cdot\frac{2n_0 d}{c}}$ (11)

Comparing eqns.9 and 11 we obtain,

$$e^{-\frac{2\pi\nu_0}{Q}\cdot\frac{2n_0 d}{c}} = R_1R_2e^{-2\alpha_c d}$$

i.e. $$e^{2\alpha_c d - \frac{4\pi\nu_0 n_0 d}{Qc}} = R_1R_2$$

Taking the natural logarithm,

$$2\alpha_c d - \frac{4\pi\nu_0 n_0 d}{Qc} = \ln(R_1R_2)$$

i.e. $$\frac{4\pi\nu_0 n_0 d}{Qc} = 2\alpha_c d - \ln(R_1R_2)$$

i.e. $$Q = \frac{4\pi\nu_0 n_0 d}{c}\left(\frac{1}{2\alpha_c d - \ln(R_1R_2)}\right) \tag{12}$$

Let $$\kappa = 2\alpha_c d - \ln(R_1R_2) = \frac{4\pi\nu_0 n_0 d}{Qc} \tag{13}$$

Using in eqn.11 we get,

$$W(t) = W_0 e^{-\frac{2\pi\nu_0}{Q}\cdot\frac{2n_0 d}{c}} = W_0 e^{-\kappa}$$

Fractional loss per round trip,

$$x = \frac{W_0 - W_0 e^{-\kappa}}{W_0} = 1 - e^{-\kappa}$$

$$e^{-\kappa} = 1 - x$$

$$-\kappa = \ln(1-x)$$

$$\kappa = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right) \tag{14}$$

By eqn.4, Passive cavity lifetime, $t_c = \frac{Q}{\omega_0} = \frac{Q}{2\pi\nu_0}$

Using eqn.12, $$t_c = \frac{2n_0 d}{c}\left(\frac{1}{2\alpha_c d - \ln(R_1R_2)}\right) \tag{15a}$$

Using eqn.13, $$= \frac{2n_0 d}{c\kappa} \tag{15b}$$

Using eqn.14 $$= \frac{2n_0 d}{c\ln\left(\frac{1}{1-x}\right)} \tag{15c}$$

By eqn.8,Full width at the half maximum (FWHM) for passive cavity, (using eqn.12)

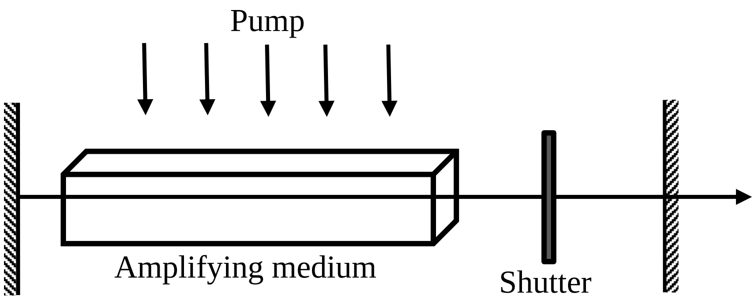
$$\Delta\nu_p = \frac{\nu_0}{Q} = \frac{c}{2\pi n_0 d}\left\{\alpha_c d - \frac{1}{2}\ln(R_1R_2)\right\} \tag{16}$$

For a typical cavity, d = 100 cm, $\alpha_c d - \frac{1}{2}\ln(R_1R_2) \sim 2\times10^{-2}$ and assuming $n_0 \approx 1$, we get,

$$\Delta\nu_p \approx 1 \text{ MHz} \tag{17}$$

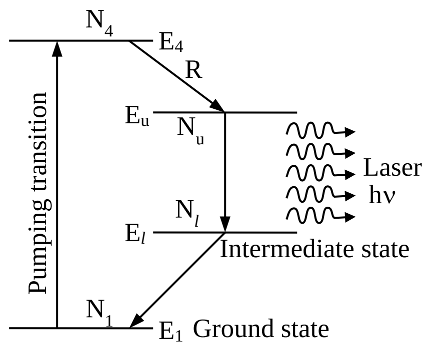
1.8 Q Switching

Q-switching is a standard technique for the pulsed operation of a laser. It is used to generate pulses of high energy but nominal pulse width in the nanosecond regime. We have seen that the quality factor of the laser cavity is determined by the losses suffered by the modes of the cavity. The smaller the losses the higher is the Q value. (Refer eqn.1 sec 1.7). Let us consider a case in which a shutter is used in front of one of the mirrors. First the shutter is closed and the medium is continuously pumped. The population in the cavity goes on increasing and reaches a high value. Now the shutter is suddenly opened. Since the inversion would correspond to a value much above the threshold the energy stored in the cavity will be released in the form of a short pulse of light with a high peak value of intensity. Because of the opening of the shutter increases the Q value from very small value before opening to a very large value after opening, this technique of producing a short intense pulse of light is referred to as *Q switching*. Two cases may arise. (1) If the shutter is opened in a time much shorter than the time required for the building up of laser oscillation, the output would consist of a giant pulse of light. (2) If the shutter opening is slow, the output would consist of a series of pulses having smaller peak power.



Theory of Q-switching: We now develop a theoretical basis for describing the time dependence of the population inversion, and also of the output pulse duration, for various inversion densities above threshold before the Q-switch is activated. Our aim is to obtain an expression for the total number of photons 'n' within the laser cavity at the laser frequency ν, and also for the total population difference $\Delta N = N_u - N_l$ within the laser gain medium at any instant during the Q-switching procedure.

Let us consider the two levels involved in the laser transition in a four-level laser (or, three-level laser). Let they are designated as u and *l*; u for upper and *l* for lower. (For a four-level laser transition from level 3 to 2 and for three-level laser it is from 2 to 1). We assume that the lower laser level (*l*) has very fast relaxation rate to lower levels (lower than the levels designated as *l*), so that it is essentially unpopulated. We also assume that only one mode has sufficient gain to oscillate and that the line is homogeneously broadened so that the same induced rate applies to all the atoms.



Let R represents the number of atoms that are being pumped into the upper level per unit time per unit volume. If the population density of the upper laser level is N_u, then the number of atoms undergoing stimulated emission from the upper laser level to the lower laser level per unit time is given by, (refer eqn.16 sec.1.2.1),

$$F_{ul} = \Gamma_{ul} V = N_u \frac{\pi^2 c^3}{\hbar \omega^3 \mu_0^3 t_{sp}} u\, g(\omega) V \tag{1}$$

where, u is the energy density of the radiation at the oscillating mode frequency ω, V is the volume of the active medium and μ_0 is its refractive index. If n is the number of photons in the cavity, we can write,

$$uV = n\hbar\omega$$

Or, $$u = \frac{n\hbar\omega}{V} \qquad (2)$$

Then eqn.1 becomes, $$F_{ul} = N_u \frac{\pi^2 c^3}{\hbar\omega^3\mu_0^3 t_{sp}} \frac{n\hbar\omega}{V} g(\omega)V$$

$$= \frac{\pi^2 c^3 g(\omega)}{\omega^2\mu_0^3 t_{sp}} nN_u = KnN_u \qquad (3)$$

where, $$K = \frac{\pi^2 c^3 g(\omega)}{\omega^2\mu_0^3 t_{sp}} = \frac{\pi^2 c^3 g(\omega) A_{ul}}{\omega^2\mu_0^3} \qquad (4)$$

The spontaneous relaxation rate (radiating and non-radiating) from the upper level to the lower level per unit volume is given by, (refer eqn.3 sec.1.2),

$$\frac{dN_u}{dt} = T_{ul}N_u$$

For the whole volume,

$$\left.\frac{d(N_uV)}{dt}\right|_{Spontaneous} = T_{ul}N_uV = (A_{ul} + S_{ul})N_uV \qquad (5)$$

Since the number of atoms in the upper level changes due to pumping, induced transitions between upper and lower levels (from upper to lower level and lower to upper level) and the spontaneous transitions from the upper to lower level, the rate of change of population of upper level is,

$$\frac{d(N_uV)}{dt} = RV + F_{lu} - F_{ul} - T_{ul}N_uV$$

$$= RV + KnN_l - KnN_u - (A_{ul} + S_{ul})N_uV \qquad (6)$$

Since the Q-switched pulse is of a very short duration we will neglect the effect of pump R and the spontaneous emission during the generation of the Q-switched pulse. It must, at the same time, be noted that for the start of the laser oscillation, the spontaneous emission is essential. Then eqn.6 becomes,

$$\frac{d(N_uV)}{dt} = -Kn(N_u - N_l) = -\frac{Kn}{V}(N_u - N_l)V$$

i.e. $$\frac{d(N_u')}{dt} = -\left(\frac{Kn}{V}\right)\Delta N' \qquad (7)$$

where, $$N_u' = N_uV \quad \text{and} \quad \Delta N' = (N_u - N_l)V \qquad (8)$$

[In Sylfvast instead of $\Delta N'$ the symbol M is used].

Similarly, the rate change of population of the lower level due to induced transitions is,

$$\frac{d(N_l')}{dt} = \left(\frac{Kn}{V}\right)\Delta N' \qquad (9)$$

Subtracting eqn.9 from eqn.7, we get,

$$\frac{d(\Delta N')}{dt} = -2\left(\frac{Kn}{V}\right)\Delta N' \qquad (10)$$

The rate of change of photon number in the cavity depends on the following four factors.

1. Increase of photons by the stimulated emission from upper level to lower level. Since every stimulated transition from upper to lower level creates a photon, the rate of increase of photons n in the cavity is given by,

$$-\frac{d(N_u V)}{dt} = KnN_u \quad (11)$$

2. Decrease of photons by the stimulated absorption from the lower level to the upper level. Due to the induced absorption the photon number in the cavity is decreased as per,

$$\frac{d(N_u V)}{dt} = KnN_l \quad (12)$$

3. Decrease of photons by the finite cavity lifetime (due to passive cavity losses). The energy of the beam bouncing back and forth in the medium decreases due to passive cavity losses. Thus, when no gain is present (no laser transition) the energy of the medium decay according to the relationship,

$$\frac{dE}{dt} = -\frac{E}{t_c} \quad ; \text{Or,} \quad E = E_0 e^{-\frac{t}{t_c}} \quad (13)$$

where, t_c is the passive cavity lifetime given by, $t_c = \frac{Q}{\omega} = \frac{Q}{2\pi\nu}$ (14)

Here Q is the quality factor. For a passive cavity resonator, we have by eqn.13 sec.1.7

$$2\alpha_c d - \ln(R_1 R_2) = \frac{4\pi\nu_0\mu_0 d}{Qc} = \frac{2\mu_0 d}{ct_c}$$

$$\text{Or,} \quad t_c = \frac{2\mu_0 d}{c}\left\{\frac{1}{2\alpha_c d - \ln(R_1 R_2)}\right\} = \frac{\mu_0 d}{c}\left\{\frac{1}{\alpha_c d - \ln(R_1 R_2)^{1/2}}\right\} \quad (15)$$

Since there are n photons in the cavity, we have,

$$E = nh\nu$$

Hence eqn.13 becomes,

$$\frac{dn}{dt} = -\frac{n}{t_c} \quad (16)$$

4. Increase of photons due to spontaneous emission from upper to lower level. It is equal to KN_u (19)

Thus, the total rate of change of n is given by adding eqns.11, 12, 16 and 19.

$$\frac{dn}{dt} = KnN_u - KnN_l - \frac{n}{t_c} + KN_u = Kn(N_u - N_l) - \frac{n}{t_c} + KN_u \quad (20a)$$

Neglecting the spontaneous emission,

$$\frac{dn}{dt} \approx \left(\frac{Kn}{V}\right)\Delta N' - \frac{n}{t_c} \quad (20b)$$

In the threshold case, $\frac{dn}{dt} = 0$, the gain is equal to the cavity losses. Then from eqn20b we get,

$$(\Delta N')_t = \frac{V}{Kt_c} \quad ; \text{Or,} \quad \frac{K}{V} = \frac{1}{(\Delta N')_t\, t_c} \quad (21)$$

Then eqn.10 becomes,

$$\frac{d(\Delta N')}{dt} = -2\left(\frac{Kn}{V}\right)\Delta N' = -2\left(\frac{n}{(\Delta N')_t\, t_c}\right)\Delta N' \quad (22)$$

Or, $$\frac{d(\Delta N')}{d\tau} = -\frac{2n}{(\Delta N')_t}\Delta N' \tag{23}$$

where, $$\tau = \frac{t}{t_c} \tag{24}$$

Eqn.23 summarizes our statement that the decrease in population inversion is twice that of the increase in the number of photons sue to the stimulated emission.
Using eqn.21, eqn.20b becomes,

$$\frac{dn}{dt} = \left(\frac{Kn}{V}\right)\Delta N' - \frac{n}{t_c} \qquad = \frac{n}{(\Delta N')_t\, t_c}\Delta N' - \frac{n}{t_c}$$

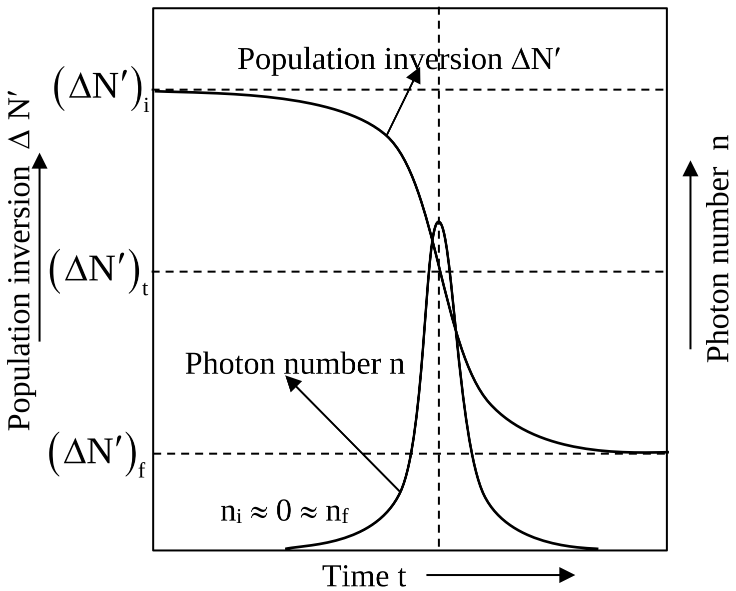
Or, $$\frac{dn}{d\tau} = n\left\{\frac{\Delta N'}{(\Delta N')_t} - 1\right\} \tag{25}$$

The term $n\frac{\Delta N'}{(\Delta N')_t}$ represents the number of photons generated within the cavity by stimulated emission per unit of normalized time. *Eqns.23 and 25 are the two principal equations in the evolution of the Q-switched pulse*. These equations give us the variation of the photon number 'n' and the population inversion $\Delta N'$ in the cavity as a function of time. These two equations are nonlinear and the solutions to the above set of equations can be obtained numerically by starting from an initial condition that at t = 0, τ = 0; $\Delta N' = (\Delta N')_i$ and $n = n_i$, where 'i' stands for the initial value. Here n_i represents the initial number of photons in the cavity generated by the spontaneous emission, which is necessary to trigger laser oscillations.

Eqn.25 can be written as,

$$\Delta N' = (\Delta N')_t + \frac{1}{n}(\Delta N')_t \frac{dn}{d\tau} \tag{26}$$

If the system is initially pumped to an inversion, $\Delta N'$ is positive. That is, $\Delta N' > \frac{1}{n}(\Delta N')_t \frac{dn}{d\tau}$ is positive. This shows that $\frac{dn}{d\tau}$ is positive, i.e. the number of photons in the cavity increases with time. The maximum number of photons appear in the cavity (n is maximum when its first derivative is zero), when $\frac{dn}{d\tau} = 0$, or, when $\Delta N' = (\Delta N')_t$. At such an instant n is very large and from eqn.23 we see that $\Delta N'$ will further reduce below $(\Delta N')_t$ and hence there is a decrease in n.



Temporal variation of population inversion ΔN′ and photon number n associated with a Q-switched pulse.

The time dependent solution of eqns.23 and 25 requires numerical computation. But we can analytically obtain the variation of n with $\Delta N'$ and from this we can draw some general conclusions regarding the peak power, the total energy in the pulse and the approximate pulse duration.

Dividing eqn.25 with eqn.23 we get,

$$\frac{dn}{d(\Delta N')} = -\frac{n\left\{\frac{\Delta N'}{(\Delta N')_t}-1\right\}}{\frac{2n}{(\Delta N')_t}\Delta N'} = \frac{1}{2}\frac{\left\{(\Delta N')_t-\Delta N'\right\}}{\Delta N'} = \frac{1}{2}\left\{\frac{(\Delta N')_t}{\Delta N'}-1\right\} \quad (27)$$

i.e.
$$dn = \frac{1}{2}\left\{(\Delta N')_t\frac{d(\Delta N')}{\Delta N'}-d(\Delta N')\right\}$$

Integrating,

$$n = \frac{1}{2}\left\{(\Delta N')_t\ln(\Delta N')-\Delta N'\right\}+C \quad (28)$$

To find the constant of integration C we apply the initial condition that when $\Delta N'$ has an initial value (when t = 0) $(\Delta N')_i$, the number of photons is n_i.

i.e.
$$n_i = \frac{1}{2}\left\{(\Delta N')_t\ln(\Delta N')_i-(\Delta N')_i\right\}+C \quad (29)$$

Subtracting eqn.29 from eqn.28 we get,

$$n-n_i = \frac{1}{2}\left\{(\Delta N')_t\ln(\Delta N')-\Delta N'\right\}+C-\frac{1}{2}\left\{(\Delta N')_t\ln(\Delta N')_i-(\Delta N')_i\right\}-C$$

$$= \frac{1}{2}\left\{(\Delta N')_t\ln\left\{\frac{(\Delta N')}{(\Delta N')_i}\right\}+(\Delta N')_i-\Delta N'\right\} \quad (30a)$$

Since the initial number of photons in the cavity (generated by spontaneous emission) is very small, eqn.30a becomes,

$$n = \frac{1}{2}\left\{(\Delta N')_t\ln\left\{\frac{(\Delta N')}{(\Delta N')_i}\right\}+(\Delta N')_i-\Delta N'\right\} \quad (30b)$$

Eqn.30a or 30b describes the *relationship between the number of photons in the cavity and the inverted population* $\Delta N' = (N_u - N_l)V$ at any particular time.

Peak power: The instantaneous power output can be approximated by multiplying the photon number by the photon energy hν and dividing by the cavity decay time t_c. That is,

$$P_{out} = \frac{nh\nu}{t_c} = \frac{h\nu}{2t_c}\left\{(\Delta N')_t\ln\left\{\frac{(\Delta N')}{(\Delta N')_i}\right\}+(\Delta N')_i-\Delta N'\right\} \quad (31a)$$

The peak power output will correspond to maximum n, which occurs when $\Delta N' = (\Delta N')_t$. Thus,

$$P_{max} = \frac{n_{max}h\nu}{t_c}$$

Using eqn.30b, with $\Delta N' = (\Delta N')_t$

$$P_{max} = \frac{h\nu}{2t_c}\left[(\Delta N')_t\ln\left\{\frac{(\Delta N')_t}{(\Delta N')_i}\right\}+(\Delta N')_i-(\Delta N')_t\right] \quad (31b)$$

Eqn.31b shows that the peak power is inversely proportional to the cavity lifetime.

Total energy: By eqn.23 we have,

$$\frac{d(\Delta N')}{d\tau} = -\frac{2n}{(\Delta N')_t}\Delta N'$$

i.e.
$$\frac{\Delta N'}{(\Delta N')_t} = -\frac{1}{2n}\frac{d(\Delta N')}{d\tau}$$

Substituting in eqn.25 we get,

$$\frac{dn}{d\tau} = n\left\{-\frac{1}{2n}\frac{d(\Delta N')}{d\tau} - 1\right\} = -\frac{1}{2}\frac{d(\Delta N')}{d\tau} - n$$

i.e.
$$dn = -\frac{1}{2}\frac{d(\Delta N')}{d\tau}d\tau - n d\tau$$

Integrating this equation from t = 0 to t = ∞ (i.e. τ = 0 to τ = ∞) we get,

$$\int_{n_i}^{n_f} dn = -\frac{1}{2}\int_0^\infty \frac{d(\Delta N')}{d\tau}d\tau - \int_0^\infty n d\tau$$

i.e.
$$n_f - n_i = -\frac{1}{2}\left[(\Delta N')_f - (\Delta N')_i\right] - \int_0^\infty n d\tau$$

i.e.
$$\int_0^\infty n d\tau = \frac{1}{2}\left\{(\Delta N')_i - (\Delta N')_f\right\} - (n_f - n_i) \qquad (32a)$$

Since the initial (before pumping) photon number n_i and the final (after Q-switching) photon number n_f are very small compared with the total integrated number of photons, we can neglect them. Then eqn.32 becomes,

$$\int_0^\infty n d\tau = \frac{1}{2}\left\{(\Delta N')_i - (\Delta N')_f\right\} \qquad (32b)$$

The total energy of the Q-switched pulse is obtained by integrating the instantaneous power output. That is, using eqn.31a

$$E = P_{out}dt = \int_0^\infty \frac{nh\nu}{t_c}dt = h\nu\int_0^\infty n d\left(\frac{t}{t_c}\right) = h\nu\int_0^\infty n d\tau$$

Using eqn.32b,
$$= \frac{h\nu}{2}\left\{(\Delta N')_i - (\Delta N')_f\right\} \qquad (33)$$

Pulse duration: An approximate estimate of the duration of the Q-switched pulse can be obtained by dividing the total energy by the peak power. That is,

$$t_d = \frac{E}{P_{max}} = \frac{\frac{h\nu}{2}\left\{(\Delta N')_i - (\Delta N')_f\right\}}{\frac{h\nu}{2t_c}\left\{(\Delta N')_t \ln\left\{\frac{(\Delta N')_t}{(\Delta N')_i}\right\} + (\Delta N')_i - (\Delta N')_t\right\}}$$

$$= t_c\left[\frac{(\Delta N')_i - (\Delta N')_f}{(\Delta N')_t \ln\left\{\frac{(\Delta N')_t}{(\Delta N')_i}\right\} + (\Delta N')_i - (\Delta N')_t}\right] \qquad (34)$$

In the above formulas $(\Delta N')_f$ is the final population inversion. In order to obtain this, we may use eqn.30a for $t \to \infty$. When $t \to \infty$, $n \to n_f$. Thus by eqn.30a, we get,

$$n_f - n_i = \frac{1}{2}\left\{(\Delta N')_t \ln\left\{\frac{(\Delta N')_f}{(\Delta N')_i}\right\} + (\Delta N')_i - (\Delta N')_f\right\} \tag{35}$$

Since the initial number of photons n_i and the final number of photons n_f are very small we can write the LHS of the eqn.35 equal to zero. Thus, for $t \to \infty$,

$$(\Delta N')_f - (\Delta N')_i = (\Delta N')_t \ln\left\{\frac{(\Delta N')_f}{(\Delta N')_i}\right\} \tag{36a}$$

Or,

$$(\Delta N')_i - (\Delta N')_f = (\Delta N')_t \ln\left\{\frac{(\Delta N')_i}{(\Delta N')_f}\right\} \tag{36b}$$

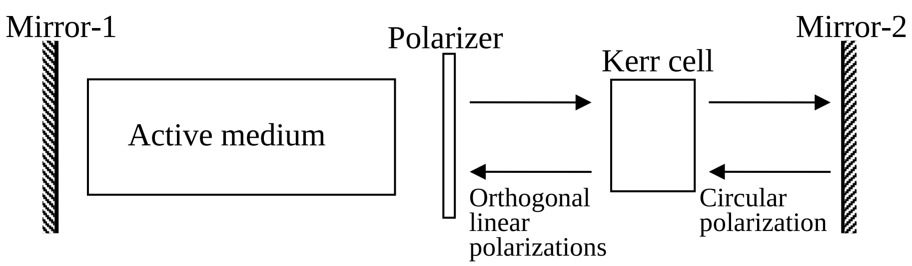
from which we can obtain $(\Delta N')_f$ for a given set of $(\Delta N')_i$ and $(\Delta N')_t$.

Different techniques for Q switching

Now we briefly discuss some Q switching techniques such as,

(a) Mechanical method: This method consists of a mechanical rotation of one of the mirrors of the laser system about an axis perpendicular to the resonator axis. When the mirrors are not parallel, the losses in the resonator are large and the pump increases the inversion beyond the threshold corresponding to the case when the mirrors are parallel. If the timing of the pump pulse is adjusted such that the laser rod is got excited to a steady maximum population inversion as the two mirrors are getting parallel, as soon as the mirrors become parallel, a giant pulse would appear at the output. Typical rotation speeds are 30,000 revolutions per minute. Since the mechanical switching is comparatively slow, one usually obtains pulse lengths of 25 to 50 nanosecond.

(b) Electronic switching technique using the Kerr and Pockels effects: Another method is the faster electronic switching techniques such as that using Kerr and Pockels effects. [If the refractive index of the electro-optic material changes linearly with the applied electric field it is termed Pockels effect. If the dependence is quadratic it is termed Kerr effect]. In this method a polarizer and a birefringent material are placed in front of one of the mirrors. [Birefringence is an optical property of certain materials, which have refractive indices that depend on polarization and propagation direction of light. Anisotropic materials show birefringence]. The two are adjusted such that the Kerr cell rotates the linearly polarized light through 90° after two



traversals of the light through the cell and is blocked by the polarizer. [First the Kerr cell changes the linearly polarized light into circularly polarized. This circularly polarized light after reflection again passes through the cell, which changes the circularly polarized light into linearly polarized in the orthogonal direction]. That is, there is no re-entry of light into the active medium. In such a position the losses in the resonator are large and the pump increases the inversion beyond the threshold value. On removing the applied voltage on the Kerr cell, the cell loses its birefringence and hence it does not rotate the polarization. In this position the losses are small, corresponds to an open shutter, and intense pulse of light appear at the output. Typical values of voltage required for the operation are a few kilovolts for a Pockels cell and a few tens of kilovolts for a Kerr cell.

(c) Method using saturable absorbers: Q switching can also be attained by saturable absorbers. These absorbers have light intensity dependent transmittance, which remains constant at small powers and begins to increase at sufficiently high intensity. As the transmittance increases the absorption coefficient decreases. The saturation intensity (which is the intensity required to reduce the absorption coefficient to one half the low power value) for normal dye solutions is about 10^7 watt/cm^2. The operation of such a device may be understood as follows. The saturable absorber is placed inside the cavity. At low powers losses by absorption is large and no laser oscillation takes place. As the pumping increases the power level inside the cavity goes on increasing and the dye begins to be bleached. This bleaching results a larger transmittance which in turn increases the power level inside the cavity. The increased power results a larger bleaching and thus the dye becomes almost transparent. At this stage the population inversion is much more than the threshold value. That is the gain is much more than the losses and thus a giant pulse is produced.

1.9 Theory of Mode locking in lasers

There are many uses of very short duration laser pulses in various fields like digital communication, diagnostics of ultra-fast processes and ablation of materials without causing significant heating of the material. In the previous section we described the process of Q-switching, which produces very high energy pulses. However, such Q-switched pulses are

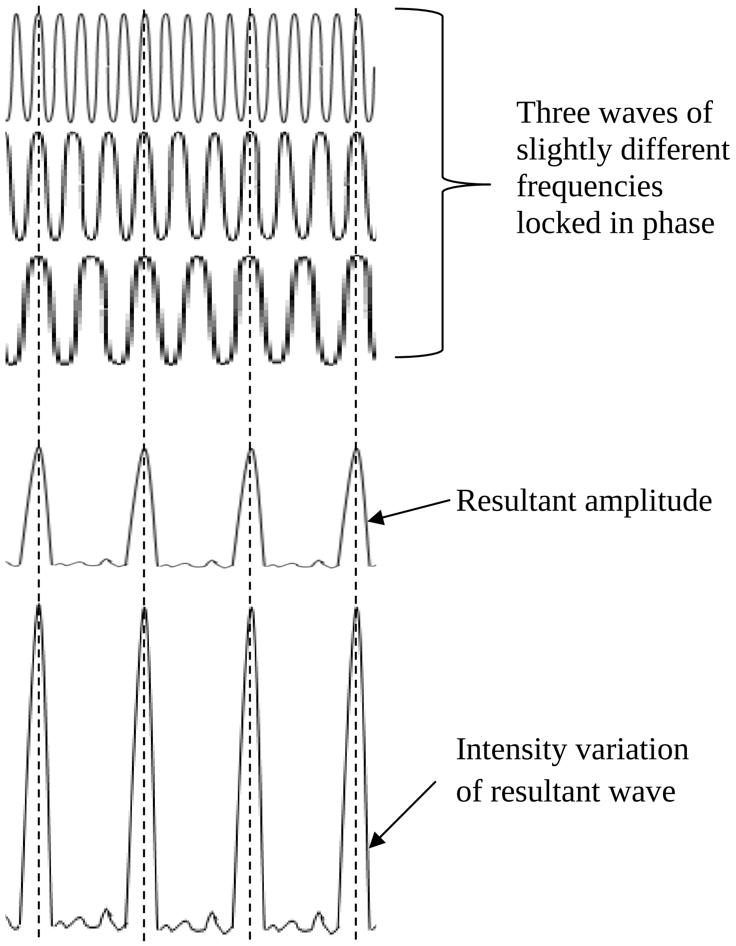


Fig.a

limited to minimum pulse durations of a few nanoseconds. Another technique, known as mode locking, has allowed the generation of optical pulses as short as 5 fs (femtosecond = 10^{-15}s).

Mode locking is achieved by combining in phase a number of distinct modes of a laser, all having slightly different frequencies. In such a case the output from the laser would be a repetitive series of pulses of light as shown in fig.d. Such a pulse train is called mode locked pulse train and this phenomenon is called mode locking. Mode locking is very similar to the case of diffraction of light from a grating. In the case of diffraction by a grating, the angular width of any of the diffracted order depends on the number of slits in the grating. Similar to that the temporal width of the mode locked pulse train depends on the number of modes that are locked in phase. (Refer fig.a).

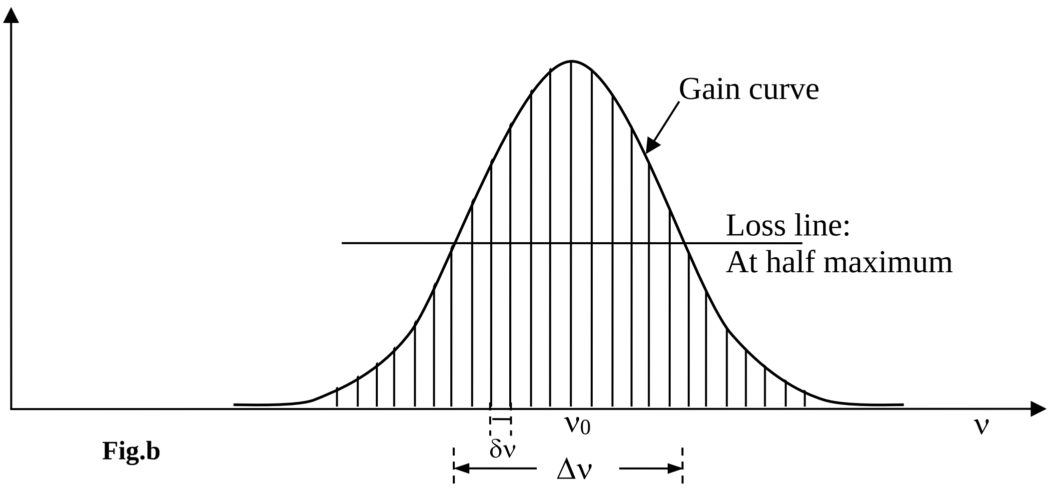


Fig.b

To understand this let us consider a laser system formed by two parallel mirrors separated by a distance ‘d’ enclosing an active medium. It has a line width Δν about a central frequency ν_0 as shown in the gain profile given above (fig.b). The frequency spacing $\Delta\nu$ of the longitudinal modes of the resonator is approximately $\frac{v}{2d} = \frac{c}{2\mu_0 d}$. [Refer eqn.21 sec.1.10.1, or, referring any book on electrodynamics we can see that the resonant frequency ‘ν’ of a rectangular cavity resonator of dimensions a, b and d (d along the z-direction) is given by,

$$\frac{4\nu^2}{v^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{q^2}{d^2}$$; where, m, n, q are integers which gives the different mode.

$$\frac{2\nu}{v} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{q^2}{d^2}\right)^{1/2} = \frac{q}{d}\left\{1 + \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\frac{d^2}{q^2}\right\}^{1/2} \approx \frac{q}{d}\left\{1 + \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\frac{d^2}{2q^2}\right\}$$

To calculate the frequency spacing we assume that m and n remain the same and q becomes q±1, then,

$$\frac{2\nu'}{v} = \left(\frac{q+1}{d}\right)\left\{1 + \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\frac{d^2}{(q+1)^2}\right\}^{1/2} \approx \left(\frac{q+1}{d}\right)\left\{1 + \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\frac{d^2}{2(q+1)^2}\right\}$$

$$\frac{2\nu'}{v} - \frac{2\nu}{v} \approx \frac{1}{d} \quad ; \quad \text{Or, frequency spacing, } \delta\nu = \nu' - \nu \approx \frac{v}{2d}$$]

If the laser medium is able to provide a net gain over a bandwidth Δν, then the laser would oscillate in a number of frequencies separated by $\delta\nu = \frac{c}{2\mu_0 d}$. The actual number of oscillating modes is the number of frequencies within the bandwidth $\Delta\nu$ (including the central one).

Number of laser oscillating modes = 1 + N =1+ integer closest to (but less than) $\frac{\Delta\nu}{\delta\nu}$ (1)

where, 1 corresponds to the mode at the centre of the line. The total output from the laser can be written as the superposition of the fields of all frequencies within the bandwidth as,

$$E(z,t) = \sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}} A_n e^{i2\pi\nu_n\left(t-\frac{z}{c}\right)+i\phi_n} \qquad (2)$$

Since we are interested in the time dependence, we suppress the space dependent part. Thus,

$$E(t) = \sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}} A_n e^{i2\pi\nu_n t+i\phi_n} \qquad (2a)$$

where, A_n represents the amplitude of the n^{th} mode and ϕ_n its phase. The complex conjugate of this equation can be written as,

$$E^*(t) = \sum_{m=-\frac{N}{2}}^{m=\frac{N}{2}} A_m^* e^{-i2\pi\nu_m t-i\phi_m} \qquad (2b)$$

The amplitudes and phases of the various modes are arbitrary. For such a case the output laser intensity is proportional to $\left|E(z,t)\right|^2$. (The intensity can be expressed as either a function of time at a fixed point or function of space coordinates (z) at a fixed time. Following the former case), i.e. considering intensity as a function of time, we get,

$$I(t) = K\left|E(t)\right|^2 = KE(t)E^*(t)$$

$$= K\sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}} A_n e^{i2\pi\nu_n t+i\phi_n} \sum_{m=-\frac{N}{2}}^{m=\frac{N}{2}} A_m e^{-i2\pi\nu_m t-i\phi_m}$$

$$= K\sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}} \left|A_n\right|^2 + K\sum_{n\neq m}\sum_{m} A_n A_m^* e^{i2\pi t(\nu_n-\nu_m)+i(\phi_n-\phi_m)} \qquad (3)$$

where, K is the proportionality constant. The first term on the RHS of eqn.3 is the n = m case. For arbitrary values of amplitudes and phases of various modes, eqn.3 represents a fluctuating output intensity of the laser and is plotted in the fig.c.

Fig.c

Since $\frac{1}{\delta\nu} = \frac{2\mu_0 d}{c}$ is exactly the time for one round trip z changes to z + 2d. Then from eqn.3 we get,

$$I(t) = I\left(t+\frac{1}{\delta\nu}\right) \qquad (4)$$

Thus, the intensity pattern repeats itself with a periodicity $\frac{1}{\delta\nu}$ even when the modes are uncorrelated. Eqn.3 also shows that within these quasiperiodic intensity fluctuations, the shortest fluctuation occurs in a time interval that corresponds to the frequency difference between two extreme modes. The extreme modes are (from fig.a) $\nu_0+\frac{\Delta\nu}{2}$ and $\nu_0-\frac{\Delta\nu}{2}$

Frequency difference between the extreme modes $= \nu_0+\frac{\Delta\nu}{2}-\left(\nu_0-\frac{\Delta\nu}{2}\right) = \Delta\nu$

Time for shortest fluctuation, $t_f = \frac{1}{\Delta\nu} = \frac{1}{N\delta\nu}$ (5)

i.e. the inverse of the bandwidth of the laser medium. When the laser is oscillating below the threshold, the various modes are largely uncorrelated as a result of the absence of the correlation between various spontaneously emitting sources. These fluctuations become much less on passing above threshold but the different modes still remain essentially uncorrelated and the output intensity fluctuates with time. From eqn.3,

$$\text{Average intensity } I_{av} = \frac{1}{\left(2d/c\right)}\int_0^{2d/c}\left[K\sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}}|A_n|^2+K\sum_{n\neq m}\sum_{m}A_nA_m^*e^{-i2\pi\left(t-\frac{z}{c}\right)(\nu_n-\nu_m)+i(\phi_n-\phi_m)}\right]dt$$

$$= K\sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}}|A_n|^2 \qquad (6)$$

Since the second term is zero. If we assume that all the modes have the same amplitude A_0,

$$I_{av} = (N+1)KA_0^2 \qquad (7)$$

Since, there are N + 1 modes within the bandwidth.

Now let us consider a case when all the modes locked in phase (i.e. they have the same phase constant). Thus, $\phi_1 = \phi_2 = ...= \phi_n = \phi_0$. For the convenience of mathematical calculation we also assume that all of them have the same amplitude A_0. Then,

$$E(t) = A_0e^{i\phi_0}\sum_{n=-\frac{N}{2}}^{n=\frac{N}{2}}e^{i2\pi\nu_n t}$$

$$= A_0e^{i\phi_0}\left[e^{i2\pi\left(\nu_0-\frac{N}{2}\delta n\right)t}+e^{i2\pi\left(\nu_0-\frac{N}{2}\delta n+\delta n\right)t}+e^{i2\pi\left(\nu_0-\frac{N}{2}\delta n+2\delta n\right)t}+........+e^{i2\pi\nu_0 t}\right.$$
$$\left.+e^{i2\pi(\nu_0+\delta n)t}+e^{i2\pi(\nu_0+2\delta n)t}+........e^{i2\pi\left(\nu_0+\frac{N}{2}\delta n\right)t}\right]$$

$$= A_0e^{i\phi_0}e^{i2\pi\nu_0 t}\left[1+e^{i2\pi\delta nt}+e^{-i2\pi\delta nt}+e^{i2\pi2\delta nt}+e^{-i2\pi2\delta nt}+ ....\right.$$
$$\left.+e^{i2\pi\frac{N}{2}\delta nt}+e^{-i2\pi\frac{N}{2}\delta nt}\right]$$

$$= A_0e^{i\phi_0}e^{i2\pi\nu_0 t}\left[1+2\cos(2\pi\delta nt)+2\cos2(2\pi\delta nt)+..........\right.$$
$$\left.+2\cos\left\{\frac{N}{2}(2\pi\delta nt)\right\}\right]$$

$$= A_0 e^{i\phi_0} e^{i2\pi\nu_0 t} \left[\frac{\sin\left\{2\pi\left(\frac{N+1}{2}\right)\delta\nu t\right\}}{\sin\left\{2\pi\left(\frac{1}{2}\right)\delta\nu t\right\}} \right] \qquad (8)$$

The output laser intensity is proportional to $|E(t)|^2 = KE(t)E^*(t)$

The intensity can be expressed as either a function of time at a fixed point or function of space coordinates (z) at a fixed time. Following the former case, we get,

$$I(t) = KA_0^2 \left[\frac{\sin\left[2\pi\left(\frac{N+1}{2}\right)\delta\nu t\right]}{\sin\left[2\pi\left(\frac{1}{2}\right)\delta\nu t\right]} \right]^2 = I_0 \left[\frac{\sin\{\pi(N+1)\delta\nu t\}}{\sin(\pi\delta\nu t)} \right]^2 \qquad (9)$$

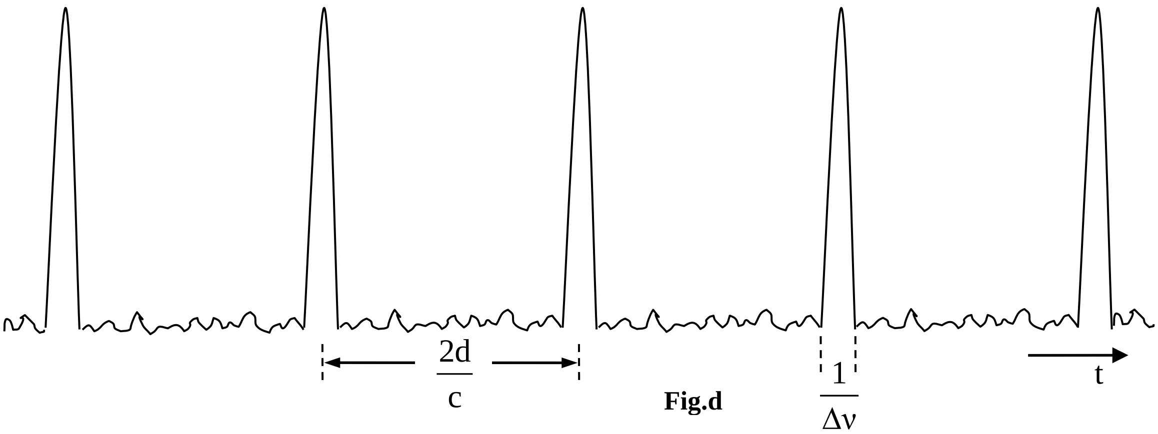


Fig.d

The intensity variation is plotted in fig.d. In this case the output is a regular sequence of well-defined pulses. From eqn.9 one may observe the following.

1. The output consists of a sequence of pulses which are separated by a time interval of $\frac{1}{\delta\nu} = \frac{2\mu_0 d}{c}$, which is exactly the time taken by the light to complete a round trip in the resonator cavity. Hence the mode-locked condition can be visualised as a pulse which is travelling back and forth in the laser cavity and which loses a part of its energy through the output mirror in every round trip.
2. Eqn.9 also gives that the duration of the pulse is approximately given by,

$$t_p \approx \frac{1}{N\delta\nu} = \frac{1}{\Delta\nu} \qquad (10)$$

   i.e. the inverse of the bandwidth of the atomic line. Thus, the larger the oscillating bandwidth of the laser medium, the smaller will be the pulse width. For typical gas lasers the pulse widths are about 1 nsec. For solid state lasers, the oscillating bandwidths are much larger and pulse widths are about 1 psec or even smaller. Such pulses are referred to as ultra-short pulses and find widespread application in the study of ultrafast phenomena in physics, chemistry and biology.
3. From eqn.9 we also obtain the peak intensity of the output pulse as,

$$I_{peak} \approx (N+1)^2 KA_0^2 = (N+1)^2 I_0 \qquad (11)$$

   This is (N + 1) times the average value given by eqn.7. Since typical solid state lasers have 10^3 to 10^4 modes of oscillation power enhancement obtained due to mode locking is very large.

Technique of active mode locking: There are different methods for mode locking. One example is active mode locking. In this technique a device is introduced into the resonator cavity which modulates periodically either the loss or the refractive index of the medium of the cavity. This technique is referred to as active mode locking since the modulating device is run by a source other than the laser. In order to understand the action of a loss modulator, we assume that the loss is modulated at a frequency equal to the intermodal spacing $\delta\nu$. As soon as the laser is switched on, the mode that lies nearest to the line centre frequency ν_0 would start oscillating first. Since the loss is modulated at the frequency $\delta\nu$, the amplitude of this mode would also oscillate at the frequency $\delta\nu$. Let the modulating wave is represented by $A_1\cos(2\pi\delta\nu t)$. Then, the resultant field is given by,

$$\begin{aligned} E &= \{A_0 + A_1\cos(2\pi\delta\nu t)\}\cos(2\pi\nu_0 t) \\ &= A_0\cos(2\pi\nu_0 t) + A_1\cos(2\pi\delta\nu t)\cos(2\pi\nu_0 t) \\ &= A_0\cos(2\pi\nu_0 t) + \frac{2A_1\cos(2\pi\delta\nu t)\cos(2\pi\nu_0 t)}{2} \\ &= A_0\cos(2\pi\nu_0 t) + \frac{A_1}{2}\cos 2\pi(\nu_0 + \delta\nu)t + \frac{A_1}{2}\cos 2\pi(\nu_0 - \delta\nu)t \qquad (12) \end{aligned}$$

The modulated central mode given by eqn.12 is the superposition of three oscillating modes with frequencies ν_0, $\nu_0 + \delta\nu$ and $\nu_0 - \delta\nu$. The oscillating fields having frequencies $\nu_0 + \delta\nu$ and $\nu_0 - \delta\nu$ force the oscillations corresponding to these frequencies into oscillation and thus these new modes have a perfect phase relationship with the mode at ν_0. Just as before, these new modes are also modulated at the frequency $\delta\nu$, which in turn create additional frequencies $\nu_0 + 2\delta\nu$ and $\nu_0 - 2\delta\nu$ in addition to those already present. This process is going on and hence all the modes are forced into oscillation in a definite phase and this leads to mode locking.

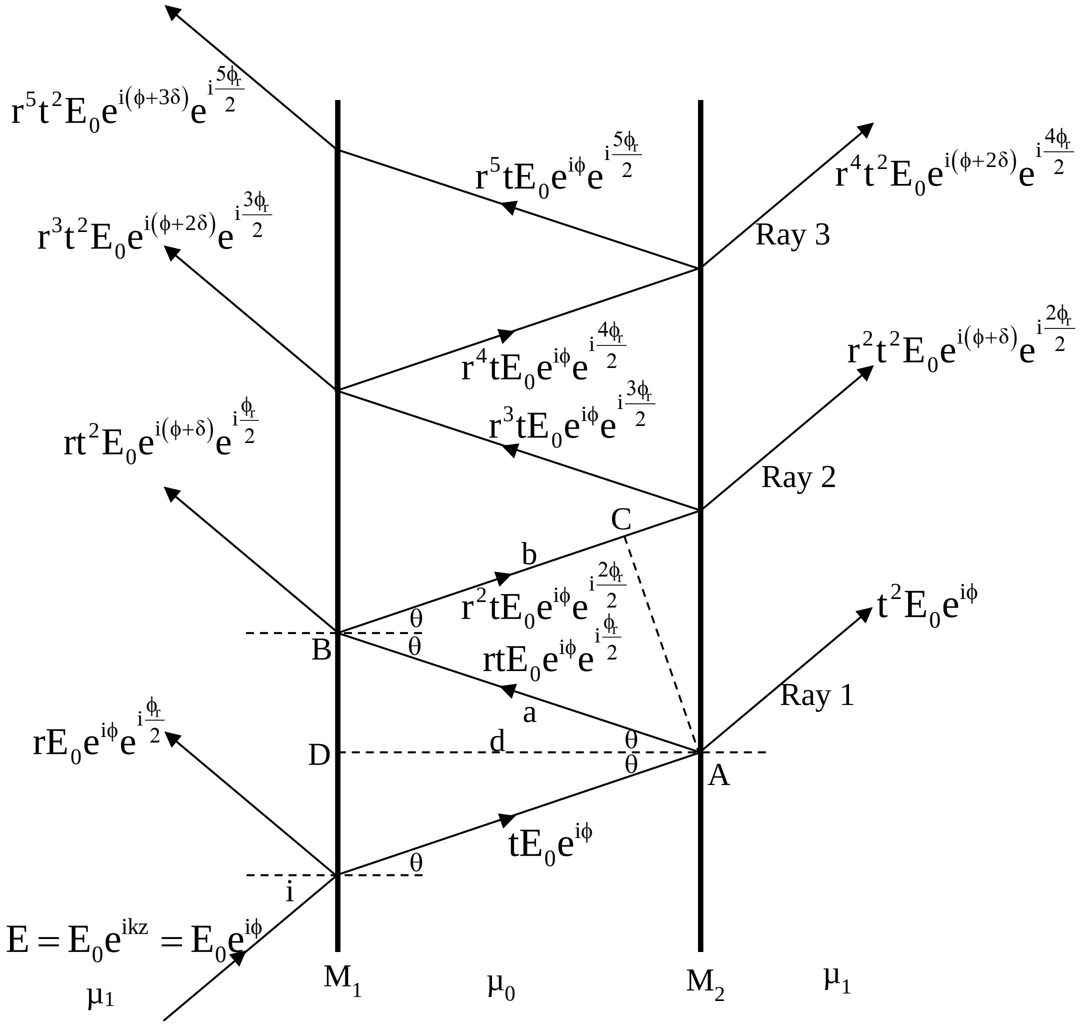
Mode locking can also be obtained by using saturable absorbers.

1.10 Laser Cavity Modes

We have already seen that in a laser system the active medium (amplifying medium) which produces light amplification is placed in between two parallel mirrors facing each other. This arrangement is known as an *optical resonator*. The region between the mirrors is known as the *cavity*. In this topic we consider the properties associated with the optical cavity of the laser. These properties play a significant role in determining the output characteristics of the laser medium. The mirrors at the ends of the cavity lead to the development of both the longitudinal modes (also known as temporal modes) and transverse modes (spatial modes).

1.10.1 Longitudinal laser cavity modes

When the mirrors are placed at the ends of the laser medium, they impose certain boundary conditions upon the electromagnetic field developed in between the mirrors. Comparing with the modes that developed in a cavity in thermal equilibrium (theory of blackbody radiation in a cavity) we can expect that similar modes may develop within the laser cavity with similar boundary condition that the electric field must be zero at the reflecting surface. To begin with our analysis, we consider the case of two-mirrored cavity, known as *Fabry-Perot resonator*, with no optical elements between the mirrors. Then we consider the effect of placing an amplifying medium between the mirrors.



Consider the multiple reflection by two mirrors M_1 and M_2 as shown in the figure above. Here the rays 1, 2, …. in the region 3 are considered as the transmitted rays and those rays on the left side as reflected rays. (In a laser cavity transmitted rays are the output laser beam). The phase of the propagating waves is given by ϕ = **k.r** = kz. The phase difference between the successive transmitted rays can be calculated as follows. Let d be the separation between the mirrors. The path difference between rays 1 and 2 is, (same as between 2 and 3, between 3 and 4 and so on)

$$AB + BC = a + b \qquad (1)$$

From the triangle ADB, $$a = \frac{d}{\cos\theta} \qquad (2)$$

From the triangle ABC, $$b = a\cos 2\theta = a\left(2\cos^2\theta - 1\right) = \frac{d}{\cos\theta}\left(2\cos^2\theta - 1\right) \qquad (3)$$

Then, the path difference between rays 1 and 2 is

$$a + b = \frac{d}{\cos\theta} + \frac{d}{\cos\theta}\left(2\cos^2\theta - 1\right)$$

$$= \frac{d}{\cos\theta}\left\{1 + \left(2\cos^2\theta - 1\right)\right\} = 2d\cos\theta \qquad (4)$$

Thus, the phase difference between the successive transmitted rays is given by,

$$\delta = \frac{2\pi}{\lambda}\left(a + b\right) = \frac{4\pi}{\lambda}d\cos\theta = 2kd\cos\theta \qquad (5)$$

Let $\frac{\phi_r}{2}$ be the phase change for one reflection. When the reflection takes place at the interface of the denser to the rarer medium $\frac{\phi_r}{2}=0$ and for rarer to denser medium $\frac{\phi_r}{2}=\pi$. Thus, the resultant amplitudes of the transmitted rays is given by,

$$\begin{aligned} E_t &= t^2E_0 + r^2 e^{i\frac{2\phi_r}{2}} t^2E_0e^{i\delta} + r^4 e^{i\frac{4\phi_r}{2}} t^2E_0e^{i2\delta} + ... \\ &= t^2E_0\left(1+r^2e^{i(\delta+\phi_r)} + r^4e^{i2(\delta+\phi_r)} + ...\right) \\ &= t^2E_0\left(1+r^2e^{i\psi}+r^4e^{i2\psi}+........\right) \quad = \quad t^2E_0\sum_{n=0}^{\infty} r^{2n}e^{in\psi} \qquad (6) \end{aligned}$$

where, $\psi=\delta+\phi_r$ (7)

Using the expansion,

$$\frac{1}{1-x}=(1-x)^{-1} = 1+x+x^2+x^3+.......$$

$$E_t = \frac{t^2E_0}{1-r^2e^{i\psi}} \qquad (8)$$

Thus, the transmitted intensity,

$$I_t = |E_t|^2 = \frac{t^4E_0^2}{\left(1-r^2e^{i\psi}\right)\left(1-r^2e^{-i\psi}\right)} = \frac{T^2E_0^2}{\left(1-Re^{i\psi}\right)\left(1-Re^{-i\psi}\right)} \qquad (9)$$

where, Reflectance, R = $|r|^2 = r^2$ and Transmittance, $T = |t|^2 = t^2$ (10)

(Here, for transmission there is no phase change and for reflection the phase change is $\frac{\phi_r}{2}$).

Eqn.9 can be written as,

$$\begin{aligned} I_t &= I_0T^2\left(\frac{1}{1-Re^{i\psi}-Re^{-i\psi}+R^2}\right) = I_0T^2\left(\frac{1}{1-2R\cos\psi+R^2}\right) \\ &= I_0T^2\left(\frac{1}{1-2R+R^2-2R\cos\psi+2R}\right) \\ &= I_0T^2\left\{\frac{1}{(1-R)^2+2R(1-\cos\psi)}\right\} = \frac{I_0T^2}{(1-R)^2}\left\{\frac{1}{1+\frac{4R\sin^2\left(\frac{\psi}{2}\right)}{(1-R)^2}}\right\} \\ &= \frac{I_0T^2}{(1-R)^2}\left\{\frac{1}{1+F'\sin^2\left(\frac{\psi}{2}\right)}\right\} \qquad (11) \end{aligned}$$

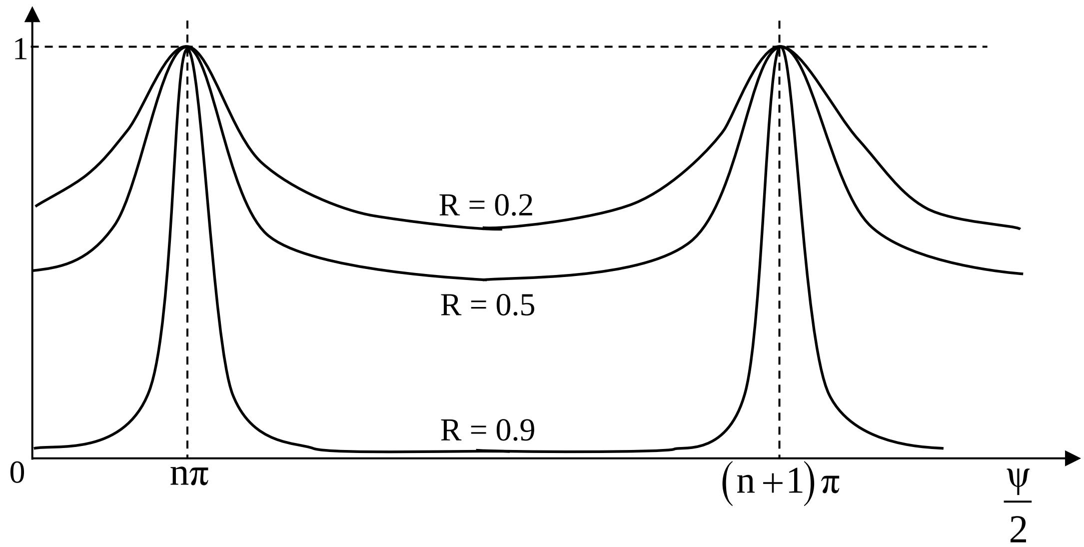
where, $$F' = \frac{4R}{(1-R)^2} \qquad (12)$$

Here the quantity in the brace is referred to as the Airy function. If there is no absorption, $R+T=1$, or, $T=1-R$, then eqn.11 becomes,

$$I_t = \frac{I_0}{1+F'\sin^2\left(\frac{\psi}{2}\right)}$$

i.e.
$$\frac{I_t}{I_0} = \frac{1}{1+F'\sin^2\left(\frac{\psi}{2}\right)} \qquad (13)$$

That is, the ratio of the transmitted intensity to the incident intensity is simply the Airy function. A plot of $\frac{I_t}{I_0}$ versus $\frac{\psi}{2}$ for different values of R is shown in the figure below.



The Airy function for a particular value of R has a maximum value of unity when its denominator is minimum, i.e. when $\frac{\psi}{2} = n\pi$; where n = 0, 1, 2, 3, ………. It has minimum value when $\frac{\psi}{2} = (2n+1)\frac{\pi}{2}$; for n = 0, 1, 2, 3, ………. . The minimum value of Airy function depends on the value of R also. We refer the values of ψ corresponding the maximum values of $\frac{I_t}{I_0}$ by, and also using eqns.5 and 7

$$\psi_{max} = 2n\pi = \frac{4\pi}{\lambda} d\cos\theta + \phi_r \qquad (14)$$

We get peaks for Airy function for n = 0, 1, 2, 3, ……. . All these peaks are identical in shape. From the figure it is clear that for larger values of R, i.e. R > 0.6, the graph is highly sharp so that a peak is obtained for the range of small values of $\frac{\psi}{2}$. So we can approximate $\sin\left(\frac{\psi}{2}\right) \approx \frac{\psi}{2}$. The values of $\frac{\psi}{2}$ corresponding to the half maximum are given by eqn.13,

$$\frac{1}{1+F'\left(\frac{\psi'}{2}\right)^2} = \frac{1}{2}$$

i.e.
$$\psi'_+ = \frac{2}{\sqrt{F'}} \quad ; \text{and} \quad \psi'_- = -\frac{2}{\sqrt{F'}}$$

Thus, the full width at half maximum (FWHM) is given by,

i.e. $$\text{FWHM} = \psi'_{+} - \psi'_{-} = \frac{4}{\sqrt{F'}} \qquad (15)$$

The separation between the adjacent peaks is given by, (from the figure)

$$\frac{\psi_{n+1}}{2} - \frac{\psi_n}{2} = (n+1)\pi - n\pi$$

i.e. $$\Delta\psi = \psi_{n+1} - \psi_n = 2\pi \qquad (16)$$

Now we define a function F equal to the ratio between peak separation and FWHM. That is, using eqn.12,

$$F = \frac{\Delta\psi}{\text{FWHM}} = \frac{2\pi}{\frac{4}{\sqrt{F'}}} = \frac{\pi\sqrt{F'}}{2} = \frac{\pi\sqrt{\frac{4R}{(1-R)^2}}}{2} = \frac{\pi\sqrt{R}}{1-R} \qquad (17a)$$

F is referred to as *finesse*. If the two mirrors have different reflectivities, instead of R we have to use $R = (R_1R_2)^{1/2}$, then F becomes,

$$F = \frac{\pi(R_1R_2)^{1/4}}{1-(R_1R_2)^{1/2}} \qquad (17b)$$

Now we consider the simple case of normal incidence and the reflection takes place at the interface of denser to rarer medium. In this case θ = 0 and $\phi_r = 0$. Then by eqn.14, we get,

$$\psi_{max} = 2n\pi = \frac{4\pi d}{\lambda} \qquad (18)$$

This wavelength at which a maximum occur can be referred to as, λ_n^{max} is given by,

$$\lambda_n^{max} = \frac{2d}{n} \quad ; \text{where } n = 1, 2, 3, \ldots\ldots \qquad (19)$$

Eqn.19 shows that these maxima occur at an infinite sequence of wavelengths, decreasing separation with increasing n. In terms of frequency eqn.19 becomes,

$$\frac{v}{v_n^{max}} = \frac{2d}{n}$$

Or, $$\frac{c}{\mu_0 v_n^{max}} = \frac{2d}{n}$$

i.e. $$v_n^{max} = \frac{cn}{2\mu_0 d} \qquad (20)$$

Thus, the difference between two successive frequencies is given by,

$$\delta v^{sep} = v_{n+1}^{max} - v_n^{max} = \frac{c}{2\mu_0 d} \qquad (21)$$

This theory is also applicable to lasers.

By eqn.17a, we have,

$$\text{FWHM} = \frac{\Delta\psi}{F}$$

In terms of frequency we can write this equation as,

$$\delta\nu_{FWHM} = \frac{\delta\nu^{sep}}{F} = \frac{1}{F}\left(\frac{c}{2\mu_0 d}\right)$$

Using eqn.17a,

$$= \left(\frac{1-R}{\pi\sqrt{R}}\right)\frac{c}{2\mu_0 d} \qquad (22a)$$

Mirrors with different reflectivities,

$$\delta\nu_{FWHM} = \left\{\frac{1-(R_1R_2)^{1/2}}{\pi(R_1R_2)^{1/4}}\right\}\frac{c}{2\mu_0 d} \qquad (22b)$$

The quality factor of the resonator is given by

$$Q = \frac{\text{Resonant frequency}}{\delta\nu_{FWHM}}$$

Using eqn.22a,

$$= \frac{\nu_0}{\left(\frac{1-R}{\pi\sqrt{R}}\right)\frac{c}{2\mu_0 d}} = \frac{2\pi\mu_0 d\sqrt{R}}{(1-R)c}\nu_0 \qquad (23a)$$

Mirrors with different reflectivities,

$$Q = \frac{2\pi\mu_0 d(R_1R_2)^{1/4}}{\left\{1-(R_1R_2)^{1/2}\right\}c}\nu_0 \qquad (23b)$$

Fabry-Perot cavity modes: For a Fabry-Perot cavity we have by eqn.19,

$$d = n\frac{\lambda_n^{max}}{2} \quad ; \text{where } n = 1, 2, 3, \ldots\ldots \qquad (24)$$

Eqn.24 shows that the integral multiple of half the wavelengths fit into the cavity spacing d. Each of these is a standing wave and is known as a *mode*. In terms of frequency, eqn.19 becomes,

$$\nu_n^{max} = n\frac{v}{2d} = \frac{nc}{2\mu_0 d} \qquad (25a)$$

If $\mu_0 = 1$,

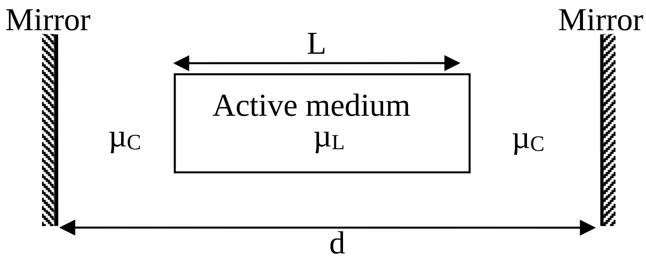
$$\nu_n^{max} = \frac{nc}{2d} \qquad (25b)$$

This shows that there are essentially an infinite number of frequencies that would fit within such a cavity. If we want to consider a wide range of frequencies, the reflectivity of the mirrors would have to be high over that entire range of frequencies.

Longitudinal laser cavity modes: A laser system is basically a Fabry-Perot cavity with an amplifying medium inserted within the cavity. So, in a laser cavity, similar modes will be set up in the form of standing wave patterns. Eqn.25 shows that these frequencies are equally spaced. The various standing waves each of a different frequency according to eqn.25 are referred to as longitudinal modes, because they are associated with longitudinal direction (along the length of the cavity) of propagation of the electromagnetic waves within the cavity.

Longitudinal mode number: We have seen that one or more longitudinal laser mode frequencies can occur when a laser gain medium is placed in between two mirrors and sufficient time is allowed for such modes to develop, typically 10 ns to 1 µs. The total number of modes is determined by the separation 'd' between the mirrors, the laser bandwidth and the type of broadening (homogeneous or inhomogeneous) that present. The mode frequencies are obtained from the eqn.25a as,

$$\nu_n = \frac{nc}{2\mu_0 d} \quad (25c)$$



This expression is valid for almost all gas lasers, solid state lasers and dye lasers in which the mirrors are placed immediately at the ends of the gain medium.

If the length 'L' of the laser medium is less than the separation 'd' between the mirrors, the different modes are given by,

$$\nu_n = \frac{nc}{2}\left\{\frac{1}{\mu_C(d-L)+\mu_L L}\right\} \quad (25d)$$

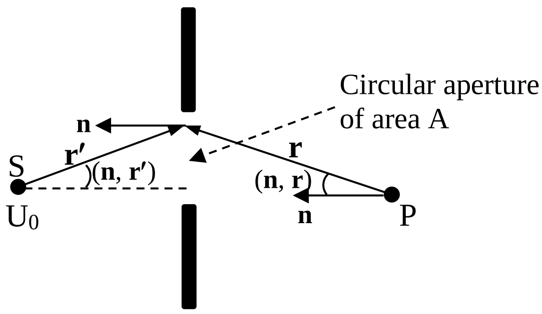
1.10.2 Transverse laser cavity modes

In the longitudinal mode we have considered two parallel mirrors of infinite extent and the light beam is normal or nearly normal to the reflecting surface. However, the laser mirrors are not of infinite extent and the beam is not plane wave. The finite size of the mirrors causes diffraction of the beam at the edges of the mirror. This leads to the loss of energy within the laser cavity.

We now consider two modifications to our previous analysis. We assume that the two mirrors are of finite size and are circular in shape. Since the diffraction effects are comparatively smaller, curved circular mirrors are more desirable for a laser cavity. We also assume that the source of light is in between the mirrors so that the incident beam is not plane wave.

In our analysis we will first obtain the expression for the transverse profile of the beam that builds up within the cavity after having undergone many reflections as the beam oscillates back and forth between the mirrors.

In our analysis we use the Huygens principle. According to this principle spherical wavefronts of amplitude U_0 are emanated from the source S and reaches the circular aperture region of area A. Then the secondary wavelets are originated from the aperture and reaches any point P at which the amplitude of the wave U_P is evaluated. Using the *Fresnel-Kirchhoff integral formula* the amplitude of the wave at P is given by,



$$U_P = -\frac{ik}{4\pi}\iint_A U_0 \frac{e^{ikr'}}{r'}\frac{e^{ikr}}{r}\{\cos(\mathbf{n},\mathbf{r})-\cos(\mathbf{n},\mathbf{r}')\}\,dA \quad (1)$$

where, dA = dxdy is the area element in the aperture region which lies in the x-y plane, **n** is a unit vector normal to the aperture plane over which the integration to be done, **r′** and **r** are respectively the vectors drawn from the source and the point P to a point in the aperture region, symbol (**n**, **r′**) is the angle between the unit vector **n** and **r′** and (**n**, **r**) is the angle between **n** and the vector **r**.

Transverse modes with plane parallel mirrors: Consider any point on the unprimed mirror as the source point. Let $U(x, y)$ is the amplitude function (amplitude distribution) at any point on the unprimed mirror, $U'(x', y')$ is the amplitude function at any point on the primed mirror and $U(x', y')$ is amplitude distribution due to the unprimed mirror evaluated at any point on the primed mirror. Then it can be shown that (avoided all the steps),

$$U'(x', y') = \gamma U(x', y') = \iint_A U(x, y) K(x, y, x', y') dxdy \qquad (2)$$

where, γ is a constant factor determined by the diffraction. The function K is known as the *kernel* of the equation and γ is the *eigenvalue* of the equation.

There are infinite number of solutions U_n and γ_n to eqn.2; each set is associated with a specific value of n, where n can take values n = 1, 2, 3, ……… These solutions correspond to the normal modes of the resonator. They are referred to as transverse modes because they represent amplitude distributions of the electromagnetic field in the transverse directions to the laser beam within the resonator.

Transverse modes with curved mirrors: An analysis similar to that of the parallel plane mirrors can be made for a laser beam developed between curved mirrors. The advantage of the curved mirrors is that the beam was focussed slightly after each reflection. This reduces the beam amplitude near the edges of the mirror and hence the diffraction loss can be reduced. The diffraction loss is related to the Fresnel number.

[The **Fresnel number** (*N*), named after the physicist Augustin-Jean Fresnel, is a dimensionless number occurring in optics, in particular in scalar diffraction theory.

For an electromagnetic wave passing through an aperture and hitting a screen, the Fresnel number *N* is defined as $N = \frac{a^2}{\lambda d}$, where, 'a' is the characteristic size (e.g. radius for spherical mirrors, for plane mirrors length and breadth of the mirrors. For rectangular mirrors two Fresnel numbers one for x-direction and the other for y direction. For square mirror they are equal.) of the aperture, 'd' is the distance of the screen from the aperture and 'λ' is the incident wavelength.

The Fresnel number is a useful concept in physical optics. Conceptually, it is the number of half-period zones in the wavefront amplitude, counted from the center to the edge of the aperture, as seen from the observation point (the center of the imaging screen), where a half-period zone is defined so that the wavefront phase changes by π when moving from one half-period zone to the next.]

The theory of confocal resonator system (radii of curvature of the mirrors is equal to the separation between the mirrors), which will be discussed in sec.1.11, shows that for a confocal resonator the different modes are designated as TEM_{npq}, where, n is the longitudinal mode number and p and q are the transverse mode numbers. When we are concerned with transverse modes the designation becomes TEM_{pq}.

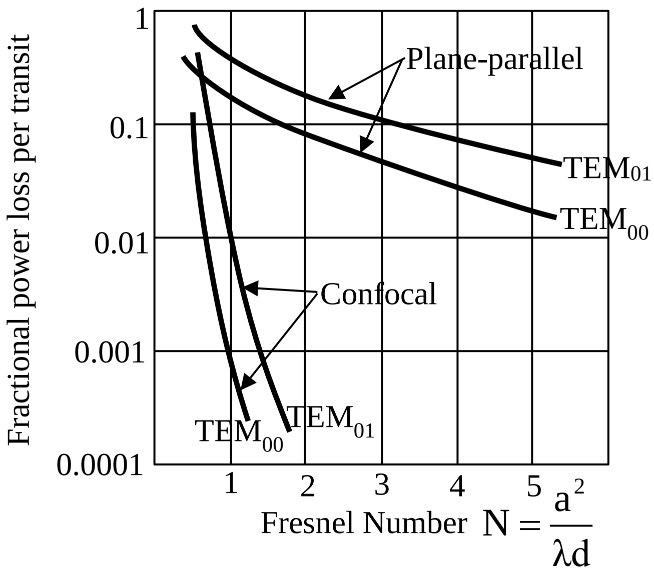


Figure gives the plot of the fractional power loss per transit versus the Fresnel number in the cases of plane parallel and confocal resonator systems for two modes TEM_{00} and TEM_{01}. In the next topic we will discuss the longitudinal and transverse modes of a confocal resonator system.

1.11 Confocal Resonator system*

Figure represents a symmetric confocal resonator system. It consists of a pair of mirrors of equal radii of curvature facing each other. They are separated by a distance equal to the radius of curvature. Our aim is to find out the modes of this symmetric resonator.

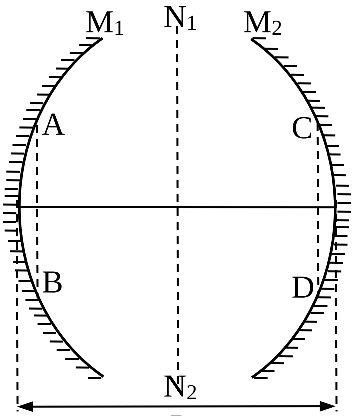


Fig.a

Since the resonator system is symmetric about the middle plane N_1N_2 the field distribution across any plane on one side of N_1N_2, say AB, after completing half a round trip (equal to the distance R; the rays from AB after reflection from mirror M_2 reach the plane CD) must repeat itself on the plane CD. The phase shift suffered by the wave in half a round trip must be an integral multiple of π. Such a condition would give us the transverse modes of the resonator. In this case to know the field distribution as it propagates through a distance z we use the Huygens principle.

Assume a coordinate system with origin at the midpoint of the plane AB and the z-axis along the axis of the resonator. Then the coordinates of the points on plane AB is (x, y, 0). Then the field distribution at the plane AB can be represented by the function f(x, y). According to Huygens principle the field distribution on another plane at a distance z (coordinates x, y, z) is given by the superposition of all fields due to the spherical waves emanating from every point on the plane at z = 0. Consider a small elemental area $dx'dy'$ centred at the point $(x', y', 0)$ on the plane at z = 0. Then the field distribution over the elemental area $dx'dy'$ is given by, $f(x', y')dx'dy'$.

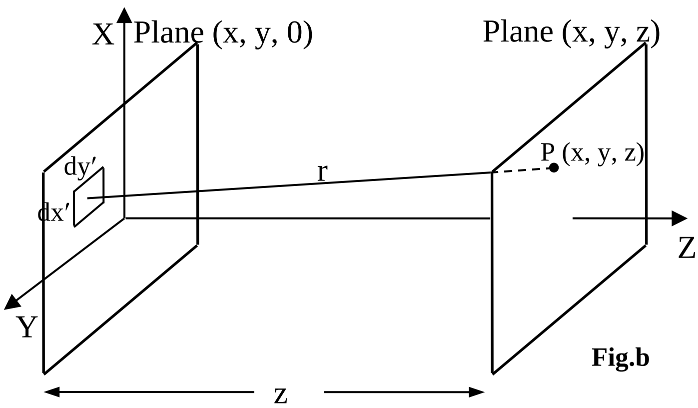


Fig.b

Since the intensity of the wave obeys inverse square law, [If I is the intensity of radiation scattered in all directions from the point O, intensity of radiation over unit area at a spherical surface of radius r is $\frac{I}{4\pi r^2}$. Then we can write $\frac{I}{4\pi r^2}dS$

$$= \frac{I}{4\pi r'^2}dS'$$

. That is the intensity is inversely proportional to r^2. Since the intensity is proportional to the square of the amplitude, the amplitude of the wave varies inversely with r], the field produced by the element $dx'dy'$ at the point P with coordinates (x, y, z) on a plane at z would be proportional to $f(x', y')\frac{e^{-ikr}}{r}dx'dy'$. Exponential factor is used for considering the phase change.

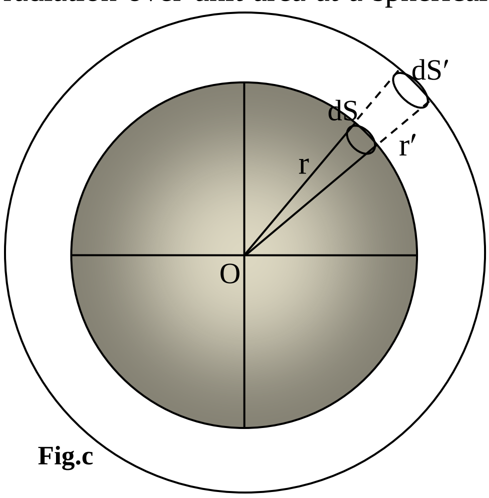


Fig.c

Distance between the points $(x', y', 0)$ and (x, y, z),

$$r = \left[(x-x')^2+(y-y')^2+z^2\right]^{1/2} \tag{1}$$

$$= z\left[1+\frac{(x-x')^2+(y-y')^2}{z^2}\right]^{1/2}$$

In the paraxial approximation we consider only the region very close to the z-axis. Then, x, y, x', y' << z. Thus we may write,

$$r \approx z\left[1+\frac{(x-x')^2+(y-y')^2}{2z^2}\right] = z+\frac{(x-x')^2+(y-y')^2}{2z} \qquad (2)$$

Total field produced (over unit area around) on any point on the plane at z by all the secondary wavelets emanated from the plane AB is given by,

$$g(x, y, z) = K\iint f(x', y')\frac{e^{-ikr}}{r}dx'dy' \qquad (3)$$

Using eqn.2,

$$= K\iint f(x', y')\frac{e^{-ik\left[z+\frac{(x-x')^2+(y-y')^2}{2z}\right]}}{\left\{z+\frac{(x-x')^2+(y-y')^2}{2z}\right\}}dx'dy'$$

$$\approx K\frac{e^{-ikz}}{z}\iint f(x', y')e^{-\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]}dx'dy' \qquad (4)$$

By a rigorous treatment one can show that $K = i/\lambda$. Eqn.4 may be used to find the field distribution on any plane.

Effect of reflection: Now we determine the effect on the field when it undergoes a reflection from a mirror of radius of curvature R. In the case of a spherical mirror, the diverging spherical wave emanating from an axial point after reflection becomes a converging spherical wave. We have the mirror equation,

$$\frac{1}{u}+\frac{1}{v} = \frac{1}{f} = \frac{2}{R} \qquad (5)$$

Thus a spherical mirror converts the incident diverging spherical wave of radius ‘u’ to a converging spherical wave of radius ‘v’. The phase change produced on the diverging spherical wave originated from ‘u’ when reaches the plane AB is given by, e^{-ikr}, where ‘r’ in this case is $r = \sqrt{x^2+y^2+u^2}$. Again in the paraxial approximation the transverse coordinates x, y << u. Then,

$$r = u\left(1+\frac{x^2+y^2}{u^2}\right)^{1/2} \approx u\left(1+\frac{x^2+y^2}{2u^2}\right) = u+\frac{x^2+y^2}{2u} \qquad (6)$$

Phase distribution on the plane AB $= e^{-ik\left(u+\frac{x^2+y^2}{2u}\right)} = e^{-iku}e^{-\frac{ik}{2u}\left(x^2+y^2\right)}$

By omitting the constant phase term e^{-iku}, we get,

Phase distribution on the plane AB $= e^{-\frac{ik}{2u}\left(x^2+y^2\right)}$

Similarly, by considering the change in direction of k and also omitting the constant phase term e^{ikv}, we get,

Phase distribution on the plane AB due to the converging wave at ‘v’ $= e^{\frac{ik}{2v}\left(x^2+y^2\right)}$

Now let p_m represents a factor which when multiplied with incident phase distribution gives the emergent (reflected) phase distribution (effect of reflection). That is,

$$p_m e^{-\frac{ik}{2u}\left(x^2+y^2\right)} = e^{\frac{ik}{2v}\left(x^2+y^2\right)}$$

$$p_m = \frac{e^{\frac{ik}{2v}\left(x^2+y^2\right)}}{e^{-\frac{ik}{2u}\left(x^2+y^2\right)}} = e^{\frac{ik}{2v}\left(x^2+y^2\right)} e^{\frac{ik}{2u}\left(x^2+y^2\right)} = e^{\frac{ik}{2}\left(x^2+y^2\right)\left(\frac{1}{u}+\frac{1}{v}\right)}$$

$$= e^{\frac{ik}{2f}\left(x^2+y^2\right)} = e^{\frac{ik}{R}\left(x^2+y^2\right)} \qquad (7)$$

Since, as the resonator system is symmetric about the middle plane N_1N_2 the field distribution across any plane on one side of N_1N_2, say AB, after completing half a round trip (equal to the distance R; the rays from AB after reflection from mirror M_2 reach the plane CD) must repeat itself on the plane CD we get from eqn.4 by considering the reflection effect (z = R constant),

$$g(x, y) = p_m K \frac{e^{-ikz}}{z} \iint f\left(x', y'\right) e^{-\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx'dy'$$

$$= \frac{i}{\lambda} \frac{e^{-ikR}}{R} \iint f\left(x', y'\right) e^{-\frac{ik}{2R}\left[(x-x')^2+(y-y')^2\right]} dx'dy' e^{\frac{ik}{R}\left(x^2+y^2\right)} \qquad (8)$$

The integration must be performed over the surface represented by the plane AB.

To determine different modes: A field distribution f(x, y) would be mode of the resonator if

$$g(x, y) = \sigma f(x, y) \qquad (9)$$

where, σ is some complex constant. The losses suffered by the field would be governed by the magnitude of σ and the phase shift of the wave is determined by the phase of σ. Then eqn.8 may be written as,

$$\sigma f(x, y) = \frac{i}{\lambda R} e^{-ikR} \iint f\left(x', y'\right) e^{-\frac{ik}{2R}\left[x^2+x'^2-2xx'+y^2+y'^2-2yy'\right]} dx'dy' e^{\frac{ik}{R}\left(x^2+y^2\right)}$$

$$= \frac{i}{\lambda R} e^{-ikR} \iint f\left(x', y'\right) e^{-\frac{ik}{2R}\left[x'^2-2xx'+y'^2-2yy'\right]} dx'dy' e^{\frac{ik}{2R}\left(x^2+y^2\right)} \qquad (10)$$

To solve the eqn.10, let

$$u(x, y) = f\left(x, y\right) e^{-\frac{ik}{2R}\left(x^2+y^2\right)} \qquad (11a)$$

Or,

$$f(x, y) = u\left(x, y\right) e^{\frac{ik}{2R}\left(x^2+y^2\right)} \qquad (11b)$$

Now we introduce two dimensionless variables,

$$\xi = \left(\frac{k}{R}\right)^{1/2} x = \left(\frac{2\pi}{\lambda R}\right)^{1/2} x \qquad (12a)$$

Or,

$$x = \left(\frac{\lambda R}{2\pi}\right)^{1/2} \xi \qquad (12b)$$

$$\eta = \left(\frac{k}{R}\right)^{1/2} y = \left(\frac{2\pi}{\lambda R}\right)^{1/2} y \qquad (12c)$$

Or,

$$y = \left(\frac{\lambda R}{2\pi}\right)^{1/2} \eta \qquad (12d)$$

Using eqns.11 and 12 in eqn.10

$$\sigma\frac{\lambda R}{2\pi}u(\xi,\eta)e^{\frac{ik}{2R}\left(\frac{\lambda R}{2\pi}\xi^2+\frac{\lambda R}{2\pi}\eta^2\right)} =$$

$$\frac{i}{\lambda R}e^{-ikR}\iint\frac{\lambda R}{2\pi}u(\xi',\eta')e^{\frac{ik}{2R}\left(\frac{\lambda R}{2\pi}\xi'^2+\frac{\lambda R}{2\pi}\eta'^2\right)}e^{-\frac{ik}{2R}\left[\frac{\lambda R}{2\pi}\xi'^2-2\frac{\lambda R}{2\pi}\xi\xi'+\frac{\lambda R}{2\pi}\eta'^2-2\frac{\lambda R}{2\pi}\eta\eta'\right]}\frac{\lambda R}{2\pi}d\xi'd\eta' e^{\frac{ik}{2R}\left(\frac{\lambda R}{2\pi}\xi^2+\frac{\lambda R}{2\pi}\eta^2\right)}$$

$$\sigma u(\xi,\eta) = \frac{i}{2\pi}e^{-ikR}\iint u(\xi',\eta')e^{i(\xi\xi'+\eta\eta')}d\xi'd\eta' \tag{13}$$

In order to simplify the analysis, we assume that the planes AB and CD are rectangular with dimensions 2a×2b. Then the limits of integration are x = –a to +a and y = –b to +b. Corresponding limits of integration of ξ are $-\xi_0$ to $+\xi_0$ and for η are $-\eta_0$ to $+\eta_0$, where,

$$\xi_0 = \left(\frac{k}{R}\right)^{1/2}a \qquad \text{and} \qquad \eta_0 = \left(\frac{k}{R}\right)^{1/2}b$$

Then eqn.13 becomes,

$$\sigma u(\xi,\eta) = \frac{i}{2\pi}e^{-ikR}\int_{-\xi_0}^{+\xi_0}\int_{-\eta_0}^{+\eta_0}u(\xi',\eta')e^{i(\xi\xi'+\eta\eta')}d\xi'd\eta' \tag{14}$$

In order to solve eqn.14 we make use of the separation of variable technique. Now we write,

$$\sigma = \kappa\tau \quad \text{and} \quad u(\xi,\eta) = p(\xi)q(\eta) \tag{15}$$

Then eqn.14 becomes,

$$\kappa\tau p(\xi)q(\eta) = \frac{i}{2\pi}e^{-ikR}\int_{-\xi_0}^{+\xi_0}\int_{-\eta_0}^{+\eta_0}p(\xi')q(\eta')e^{i\xi\xi'}e^{i\eta\eta'}d\xi'd\eta'$$

i.e. $$\kappa p(\xi)\tau q(\eta) = \left(\frac{i}{2\pi}\right)^{1/2}e^{-i\frac{kR}{2}}\int_{-\xi_0}^{+\xi_0}p(\xi')e^{i\xi\xi'}d\xi'\left(\frac{i}{2\pi}\right)^{1/2}e^{-i\frac{kR}{2}}\int_{-\eta_0}^{+\eta_0}q(\eta')e^{i\eta\eta'}d\eta'$$

Splitting this equation we get,

$$\kappa p(\xi) = \left(\frac{i}{2\pi}\right)^{1/2}e^{-i\frac{kR}{2}}\int_{-\xi_0}^{+\xi_0}p(\xi')e^{i\xi\xi'}d\xi' \tag{16}$$

And, $$\tau q(\eta) = \left(\frac{i}{2\pi}\right)^{1/2}e^{-i\frac{kR}{2}}\int_{-\eta_0}^{+\eta_0}q(\eta')e^{i\eta\eta'}d\eta' \tag{17}$$

Eqns.16 and 17 are finite Fourier transforms. In the limit $\xi_0 \to \infty$ and $\eta_0 \to \infty$, they reduce to the usual Fourier transforms. It has been shown by Slepian and Pollack (1961) that the solutions eqns.16 and 17 are prolate spheroidal functions. We now consider only resonators of large Fresnel numbers, $N_1 = \frac{a^2}{\lambda R}$ and $N_2 = \frac{b^2}{\lambda R}$, so that $\xi_0 = \left(\frac{k}{R}\right)^{1/2}a = \left(\frac{2\pi a^2}{\lambda R}\right)^{1/2} >> 1$ and $\eta_0 = \left(\frac{k}{R}\right)^{1/2}b = \left(\frac{2\pi b^2}{\lambda R}\right)^{1/2} >> 1$ and in these cases we consider the limits of integration of eqns.16 and 17 are from $-\infty$ to $+\infty$.

Eqn.16 can be written as,

$$\kappa p(\xi) = \left(\frac{i}{2\pi}\right)^{1/2} e^{-i\frac{kR}{2}} \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi'$$

$$\left(\frac{i}{2\pi}\right)^{-1/2} e^{i\frac{kR}{2}} \kappa p(\xi) = \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi'$$

i.e.

$$A p(\xi) = \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi' \quad (18)$$

where,

$$A = \left(\frac{i}{2\pi}\right)^{-1/2} e^{i\frac{kR}{2}} \kappa \quad (19)$$

We can also write an identical equation for $q(\eta)$. Eqn.18 requires that, apart from some constant factors, $p(\xi)$ be its own Fourier transform. Differentiating eqn.18 twice with respect to ξ, we get,

$$A\frac{d^2p}{d\xi^2} = -\int_{-\xi_0}^{+\xi_0} \xi'^2 p(\xi') e^{i\xi\xi'} d\xi' \quad (20)$$

We now consider the integral,

$$I = \int_{-\xi_0}^{+\xi_0} \frac{d^2p}{d\xi'^2} e^{i\xi\xi'} d\xi' \quad (21)$$

Integrating eqn.21 by parts [$\int UdV = UV - \int VdU$], we get,

$$\int_{-\xi_0}^{+\xi_0} \frac{d^2p}{d\xi'^2} e^{i\xi\xi'} d\xi' = \left[e^{i\xi\xi'}\frac{dp}{d\xi'}\right]_{-\xi_0}^{+\xi_0} - i\xi \int_{-\xi_0}^{+\xi_0} \frac{dp}{d\xi'} e^{i\xi\xi'} d\xi'$$

Integrating once again by parts,

$$= \left[e^{i\xi\xi'}\frac{dp}{d\xi'}\right]_{-\xi_0}^{+\xi_0} - i\xi\left[p(\xi')e^{i\xi\xi'}\right]_{-\xi_0}^{+\xi_0} - \xi^2 \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi'$$

For the required mode, we assume that $p(\xi)$ and its derivative vanish at infinity and using eqn.18, we get,

$$\int_{-\xi_0}^{+\xi_0} \frac{d^2p}{d\xi'^2} e^{i\xi\xi'} d\xi' = -A\xi^2 p(\xi) \quad (22)$$

Adding eqns.20 and 22, we obtain,

$$A\left\{\frac{d^2p}{d\xi^2} - \xi^2 p(\xi)\right\} = \int_{-\xi_0}^{+\xi_0} \left\{\frac{d^2p}{d\xi'^2} - \xi'^2 p\right\} e^{i\xi\xi'} d\xi' \quad (23)$$

Comparing eqns.18 and 23 we see that both $p(\xi)$ and $\left\{\frac{d^2p}{d\xi^2} - \xi^2 p\right\}$ satisfy the same equation. Then one must have,

$$\frac{d^2p}{d\xi^2} - \xi^2 p = -Kp(\xi) \quad (24)$$

where, K is a constant. Rewriting eqn.24 we get,

$$\frac{d^2p}{d\xi^2}\left(K - \xi^2\right)p = 0 \quad (25)$$

The solutions of eqn.25 with the condition that $p(\xi)$ vanish at large values of ξ are the Hermite-Gauss functions. For each choice of the parameter K there is a different function p_m. Each function consists of a polynomial $H_m(\xi)$, called Hermite polynomial, in either odd or even powers of ξ, an exponential factor $\mathbf{exp}(-\xi^2/2)$ and a numerical coefficient which is needed for p_n to meet the normalization condition,

i.e $$p_m(\xi) = N_m H_m(\xi) e^{-\xi^2/2} \qquad (26a)$$

Or, $$p_m(\xi') = N_m H_m(\xi') e^{-\xi'^2/2}$$

where, $$H_m(\xi) = (2\xi)^m - \frac{m(m-1)}{1!}(2\xi)^{m-2} + \frac{m(m-1)(m-2)(m-3)}{2!}(2\xi)^{m-4} - \ldots\ldots.$$

$$= \sum_{s=0}^{m/2} (-1)^s \frac{m!}{(m-2s)!s!} (2\xi)^{m-2s}$$

where, the summation over s is up to m/2 when m is even and (m–1)/2 when m is odd.

Some Hermite polynomials

m	$H_m(y)$
0	1
1	$2y$
2	$4y^2 - 2$
3	$8y^3 - 12y$
4	$16y^4 - 48y^2 + 12$
5	$32y^5 - 160y^3 + 120y$

(27)

Also we get a similar solution for eqn.17.

$$q_n(\eta) = N_n H_n(\eta) e^{-\eta^2/2} \qquad (26b)$$

Thus the complete solution of eqn.10 may be written as,

$$f(x, y) = N_m H_m(\xi) e^{-\xi^2/2} N_n H_n(\eta) e^{-\eta^2/2} e^{\frac{ik}{2R}(x^2+y^2)}$$

$$= C H_m(\xi) H_n(\eta) e^{-\left(\frac{\xi^2+\eta^2}{2}\right)} e^{i\left(\frac{\xi^2+\eta^2}{2}\right)} \qquad (28)$$

where, C is some constant. Here m and n represent the transverse mode numbers that determine the transverse field distribution of the mode. Hermite-Gauss function satisfies the equation,

$$i^m H_m(\xi) e^{-\xi^2/2} = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} H_m(\xi') e^{-\xi'^2/2} e^{i\xi\xi'} d\xi' \qquad (29)$$

By eqn.16

$$i^m \kappa p(\xi) = i^m \left(\frac{i}{2\pi}\right)^{1/2} e^{-i\frac{kR}{2}} \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi'$$

Using eqn.26

$$\kappa N_m i^m H_m(\xi) e^{-\xi^2/2} = i^m \left(\frac{i}{2\pi}\right)^{1/2} e^{-i\frac{kR}{2}} \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi'$$

Using eqn.29 to RHS

$$\kappa (2\pi)^{-1/2} \int_{-\infty}^{+\infty} N_m H_m(\xi') e^{-\xi'^2/2} e^{i\xi\xi'} d\xi' = i^m \left(\frac{i}{2\pi}\right)^{1/2} e^{-i\frac{kR}{2}} \int_{-\xi_0}^{+\xi_0} p(\xi') e^{i\xi\xi'} d\xi'$$

Eqn.26a can also be written as,

Or, $$p_m(\xi') = N_m H_m(\xi') e^{-\xi'^2/2}$$

Using this equation to RHS of above equation we get

$$\kappa (2\pi)^{-1/2} \int_{-\infty}^{+\infty} N_m H_m(\xi') e^{-\xi'^2/2} e^{i\xi\xi'} d\xi' = i^m \left(\frac{i}{2\pi}\right)^{1/2} e^{-i\frac{kR}{2}} \int_{-\xi_0}^{+\xi_0} N_m H_m(\xi') e^{-\xi'^2/2} e^{i\xi\xi'} d\xi'$$

Thus, $$\kappa = i^m (i)^{1/2} e^{-i\frac{kR}{2}} = i^{m+\frac{1}{2}} e^{-i\frac{kR}{2}}$$

Using, $$i = e^{i\frac{\pi}{2}}$$

Then, $$\kappa = \left(e^{i\frac{\pi}{2}}\right)^{m+\frac{1}{2}} e^{-i\frac{kR}{2}} = e^{i\left(m+\frac{1}{2}\right)\frac{\pi}{2}} e^{-i\frac{kR}{2}} = e^{-i\left[\frac{kR}{2}-\left(m+\frac{1}{2}\right)\frac{\pi}{2}\right]} \tag{30}$$

Similarly, from eqn.17,

$$\tau = i^n (i)^{1/2} e^{-i\frac{kR}{2}} = e^{-i\left[\frac{kR}{2}-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right]} \tag{31}$$

Then by eqn.15,

$$\sigma = \kappa\tau = e^{-i\left[kR-(m+n+1)\frac{\pi}{2}\right]} \tag{32}$$

We see that $|\sigma| = 1$. This implies that there is no loss in the cavity. This is because of the mirrors are assumed to have extremely large transverse dimensions. The phase shift of σ represents the phase shift suffered by the wave in half a round trip. Thus one must have,

$$kR - (m+n+1)\frac{\pi}{2} = q\pi \quad ; \quad q = 1, 2, 3, \ldots \text{ refers the longitudinal mode number} \tag{33}$$

i.e. $$\frac{2\pi\nu}{c} R - (m+n+1)\frac{\pi}{2} = q\pi$$

i.e. $$\frac{2\pi\nu}{c} R = (2q+m+n+1)\frac{\pi}{2}$$

Thus, the frequencies of oscillations in the cavity are given by,

$$\nu_{mnq} = (2q+m+n+1)\frac{c}{4R} \tag{34}$$

All the modes having the same value of $(2q+m+n+1)$ (but different q, m and n values) would have the same oscillation frequency and hence would be degenerate. If q changes to q+1, eqn.34 becomes,

$$\nu_{mn(q+1)} = \left\{2(q+1)+m+n+1\right\}\frac{c}{4R} = \left\{2q+2+m+n+1\right\}\frac{c}{4R}$$

Then the frequency separation between two modes having the same values of m and n but adjacent values of q is given by,

$$\Delta\nu_q = \nu_{mn(q+1)} - \nu_{mnq} = \frac{c}{2R} \qquad (35)$$

Similarly by changing m to m+1 or n to n+1 in eqn.34 and taking difference one may get the frequency separation between two transverse modes corresponding to the same value of q.

$$\Delta\nu_m \text{ (or } \Delta\nu_n) = \nu_{(m+1)nq} - \nu_{mnq} = \frac{c}{4R} \qquad (35)$$

which is half that between two consecutive longitudinal modes. See fig.d.

The transverse intensity distribution (square of the amplitude) corresponding to the mode amplitude distribution given by eqn.26 is depicted in fig.e.

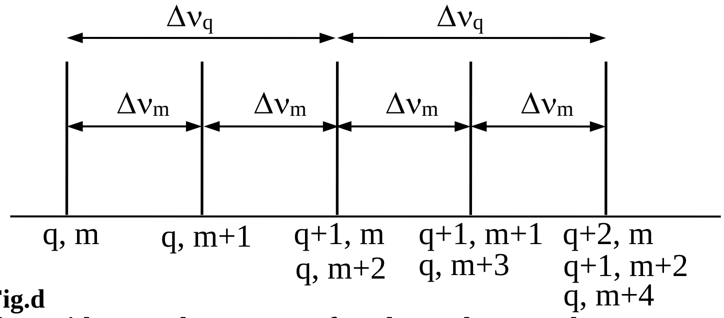


Fig.d

Lines with more than one set of mode numbers are degenerate

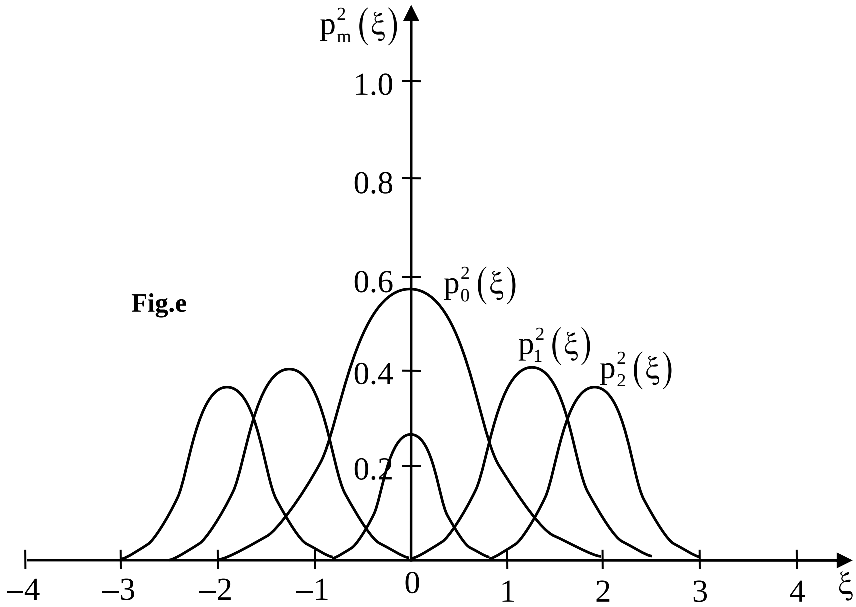


Fig.e

The field distribution given by eqn.28 corresponds to a plane passing through the pole of the mirror. It can be shown that the field distribution at a plane midway between the mirrors is,

$$f_M(x, y) = CH_m\left(\xi\sqrt{2}\right)H_n\left(\eta\sqrt{2}\right)e^{-\left(\frac{\xi^2+\eta^2}{2}\right)}e^{-i\left(\frac{kR}{2}-\frac{\pi}{2}\right)} \qquad (36)$$

Eqn.36 shows that the phase of the field given by the term $e^{-i\left(\frac{KR}{2}-\frac{\pi}{2}\right)}$ is a constant at a plane midway between the mirrors and hence the phase fronts are plane there. The phase front of other modal distribution is curved with radius of curvature R, which is equal to the radius of curvature of the resonator mirror.

1.12 Analysis of optical resonators using geometrical optics

(ABCD Matrices and condition for stability of resonators)

In this section we use the matrix method of the geometrical optics for the analysis of the general spherical resonator consisting of two mirrors of radii of curvature R_1 and R_2. Since we are using mirrors of large Fresnel numbers (see page 53) the diffraction loss is small. We assume that the resonator is stable. In a stable resonator the ray of light may keep bouncing between the mirrors indefinitely, whereas in an unstable resonator the ray may escape from the resonator after a few to and fro motions.

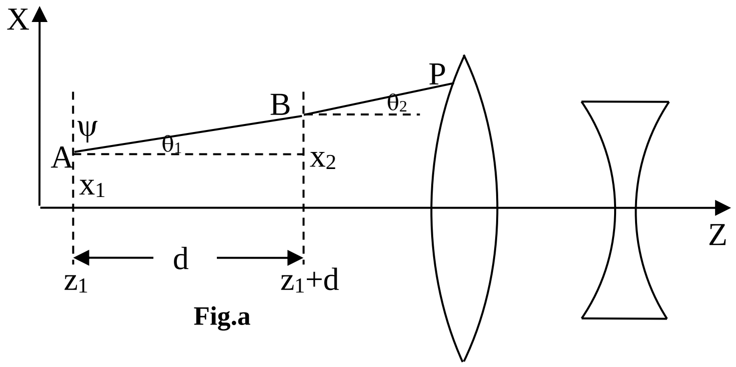


Fig.a

Consider a ray of light propagating in a homogeneous medium in the x-z plane as shown in fig.a. The ray at any point may be described by two coordinates, say x and θ, where x is the height of the ray from z-axis and θ is the angle made by the ray with the z-axis. In this analysis we restrict ourselves to only paraxial rays, i.e. the rays very close to the z-axis. This approximation is termed as *paraxial approximation*. In this approximation x and θ are very small, so that $\sin\theta = \tan\theta \approx \theta$. Let the slope of the ray at any position is $x' = \frac{dx}{dz} = \tan\theta \approx \theta$. Let x_1' and x_2' be the slopes of the ray at the positions $z = z_1$ and $z = z_1 + d$.

Translation matrix: From the fig.a,

$$x_2 = x_1 + d\tan\theta = x_1 + x_1'd \tag{1}$$

If the two positions are in the same homogeneous medium,

$$x_2' = \left.\frac{dx}{dz}\right|_{z_1+d} = \left.\frac{dx}{dz}\right|_{z_1} = x_1' \tag{2}$$

Eqns.1 and 2 may be combined into the following matrix equation,

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \tag{3}$$

The matrix $T = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ is called the *translation matrix*. (4)

Eqn.3 shows that the effect of propagation of the ray through a homogeneous medium is achieved by the translational operation with the translation matrix. Notice that the determinant of T is,

$$\det T = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} = 1 \tag{5}$$

Reflection matrix: At the point of reflection, the incident ray and the reflected ray have the same height. Thus,

$$x_2 = x_1 \tag{6}$$

The reflection produces a change in the direction and hence the slope of the ray. Thus the incident and the reflected rays have different slopes. We determine the relationships between the incident and the reflected rays using the mirror formula.

$$\frac{1}{u}+\frac{1}{v} = \frac{1}{f} = \frac{2}{R} \tag{7}$$

Slope of the incident ray, $x_1' = \frac{x_1}{u}$

Slope of the reflected ray, $x_2' = -\frac{x_1}{v}$

Then, $x_2' - x_1' = -\frac{x_1}{v} - \frac{x_1}{u} = -\frac{2x_1}{R}$

Or, $$x_2' = x_1' - \frac{2x_1}{R} \tag{8}$$

Eqns.6 and 8 can be combined to a matrix equation,

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \tag{9a}$$

Then the effect of reflection by a concave mirror can be characterized by a 2×2 matrix called the reflection matrix and is given by,

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \tag{9b}$$

It may be noted that,

$$\det \mathcal{R} = \begin{vmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{vmatrix} = 1 \tag{9c}$$

System Matrix: Consider an optical system consisting of a number of reflecting and refracting surfaces. The ray entering into the system is specified by $\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$. When it leaves the system it can be specified by $\begin{pmatrix} x_2 \\ x_2' \end{pmatrix}$. Then, in general, one can write,

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = [S] \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \tag{10}$$

where the matrix,

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{11}$$

is called the *system matrix* and is determined solely by the optical system. When the ray passes through an optical system we need only translation and refraction operations. (Reflection can be treated as a special case of refraction by choosing $n_2 = -n_1$). Hence, in general, the system matrix in the case of resonators using spherical mirrors is the product of reflection and translation matrices. Since the determinant of product matrices is the product of the determinants of the matrices, we obtain,

$$\det S \;=\; AD - BC \;=\; 1 \qquad (13)$$

Since the element 'd' of translation matrix has dimensions of length and the element $-2/R$ of the reflection matrix has dimensions of inverse length, the elements A and D are dimensionless and the product BC also is dimensionless.

To find the system matrix for a system of two facing concave mirrors: Consider a system of two concave mirrors M_1 and M_2 of radii of curvatures R_1 and R_2 respectively as shown in fig.b. Let d be the distance between them. The plane EF is at the midway between the two mirrors. Let a paraxial ray of light with coordinates $\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$ starts from the plane EF and moves towards M_2. It reaches the mirror M_2 and gets reflected at M_2. Then this ray travels towards M_1 and after undergoing a reflection at M_1 reaches again at the plane EF. Now the ray undergoes a to and fro motion or it completes a cycle of oscillation. The coordinates $\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$ of the final ray is obtained by a transformation with the system matrix. Now we take the radius of curvature of a concave mirror as positive and that of a convex mirror as negative and the distances for real objects and images as positive and that for imaginary as negative.

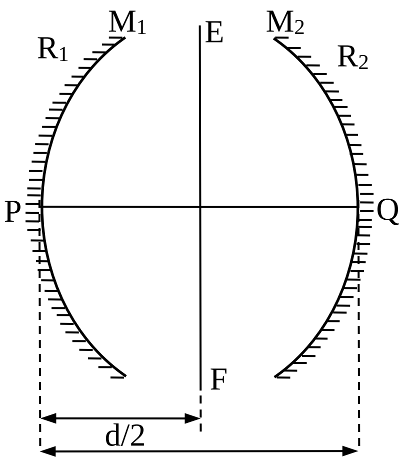


Fig.b

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{2} \\ -\frac{2}{R_2} & 1-\frac{d}{R_2} \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix}\begin{pmatrix} 1-\frac{2d}{R_2} & \frac{3d}{2}-\frac{d^2}{R_2} \\ -\frac{2}{R_2} & 1-\frac{d}{R_2} \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-\frac{2d}{R_2} & \frac{3d}{2}-\frac{d^2}{R_2} \\ -\frac{2}{R_1}-\frac{2}{R_2}+\frac{4d}{R_1R_2} & 1-\frac{d}{R_2}-\frac{3d}{R_1}+\frac{2d^2}{R_1R_2} \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{d}{R_1}-\frac{3d}{R_2}+\frac{2d^2}{R_1R_2} & 2d-\frac{3d^2}{2R_1}-\frac{3d^2}{2R_2}+\frac{d^3}{R_1R_2} \\ -\frac{2}{R_1}-\frac{2}{R_2}+\frac{4d}{R_1R_2} & 1-\frac{3d}{R_1}-\frac{d}{R_2}+\frac{2d^2}{R_1R_2} \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Thus, the system matrix for one complete traversal is given by,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-\frac{d}{R_1}-\frac{3d}{R_2}+\frac{2d^2}{R_1R_2} & 2d-\frac{3d^2}{2R_1}-\frac{3d^2}{2R_2}+\frac{d^3}{R_1R_2} \\ -\frac{2}{R_1}-\frac{2}{R_2}+\frac{4d}{R_1R_2} & 1-\frac{3d}{R_1}-\frac{d}{R_2}+\frac{2d^2}{R_1R_2} \end{pmatrix} \quad (14)$$

The ray after two complete traversals is given by,

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

The final ray after n complete traversals becomes,

$$\begin{pmatrix} x_n \\ x_n' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \quad (15)$$

For a stable resonator $\begin{pmatrix} x_n \\ x_n' \end{pmatrix}$ should not diverge after n complete traversals. In order to obtain the stability criterion we have to look at the n^{th} power of the system matrix. This can easily be done by diagonalizing the system matrix.

A matrix U is said to diagonalize a matrix S if $\quad U^{-1}SU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (16)$

where, U^{-1} is the inverse of U defined by, $\quad U^{-1}U = UU^{-1} = I \quad (17)$

The identity matrix is defined as, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

λ_1 and λ_2 in eqn.16 are called the eigen values of the matrix S. Pre multiplying by U and post multiplying by U^{-1} eqn.16 becomes,

$$UU^{-1}SUU^{-1} = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^{-1}$$

Using eqn.17 we get,

$$S = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^{-1} \quad (18)$$

Then,

$$S^n = U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^{-1}\, U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}U^{-1} \text{ .... n times}$$

$$= U\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n U^{-1} = U\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} U^{-1} \quad (19)$$

Now to find out λ_1 and λ_2 we write the eigen value equation,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

i.e. $\begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \quad (20)$

In order to exist the nontrivial solution for eqn.20 the determinant of the square matrix must vanish, then

$$\begin{vmatrix} A-\lambda & B \\ C & D-\lambda \end{vmatrix} = 0 \tag{21}$$

i.e. $(A-\lambda)(D-\lambda)-BC = 0$

i.e. $\lambda^2-(A+D)\lambda+AD-BC = 0$

Using eqn.13,

$$\lambda^2-(A+D)\lambda+1 = 0 \tag{22}$$

The two solutions of the quadratic eqn.22 are given by,

$$\lambda_1 = \frac{(A+D)+\sqrt{(A+D)^2-4}}{2} = \left(\frac{A+D}{2}\right)+\sqrt{\left(\frac{A+D}{2}\right)^2-1} \tag{23a}$$

And $$\lambda_2 = \frac{(A+D)-\sqrt{(A+D)^2-4}}{2} = \left(\frac{A+D}{2}\right)-\sqrt{\left(\frac{A+D}{2}\right)^2-1} \tag{23b}$$

Eqns.23a and 23b are the eigen values of the system matrix S. Now we write,

$$\cos\theta = \frac{A+D}{2} \tag{24}$$

Then, $$\lambda_1 = \cos\theta+i\sin\theta = e^{i\theta} \tag{25}$$

$$\lambda_2 = \cos\theta-i\sin\theta = e^{-i\theta} \tag{26}$$

From eqns.19, 25 and 26 it follows that S^n should not diverge as n increases if θ must be real and hence cosθ must be such that,

$$-1<\cos\theta<1 \quad \text{i.e.} \quad -1<\frac{A+D}{2}<1 \tag{27}$$

Eqn.27 represents the stability condition of the resonator system.

By eqn.14, $$\frac{A+D}{2} = \frac{1-\frac{d}{R_1}-\frac{3d}{R_2}+\frac{2d^2}{R_1R_2}+1-\frac{3d}{R_1}-\frac{d}{R_2}+\frac{2d^2}{R_1R_2}}{2}$$

$$= 1-\frac{2d}{R_1}-\frac{2d}{R_2}+\frac{2d^2}{R_1R_2}$$

Then the condition for stability (eqn.27) becomes, $-1\le 1-\frac{2d}{R_1}-\frac{2d}{R_2}+\frac{2d^2}{R_1R_2}\le 1$

By adding 1 throughout, $0\le 2-\frac{2d}{R_1}-\frac{2d}{R_2}+\frac{2d^2}{R_1R_2}\le 2$

By dividing throughout by 2, the condition for stability becomes,

$$0\le\left(1-\frac{d}{R_1}\right)\left(1-\frac{d}{R_2}\right)\le 1 \tag{28a}$$

Or, $$0\le g_1g_2\le 1 \tag{28b}$$

where, $$g_1=\left(1-\frac{d}{R_1}\right) \text{ and } g_2=\left(1-\frac{d}{R_2}\right) \tag{29}$$

Thus, for a resonator to be stable R_1, R_2 and d must satisfy eqn.28.

This condition can be expressed in the form of a stability diagram as shown in fig.b. The clear regions are regions where eqn.28 is not satisfied, i.e. $g_1g_2 > 1$. In this region the values of R_1, R_2 and d are such that the condition given by eqn.28 is not satisfied and the cavity is unstable. For the shaded regions the condition for stability is satisfied and the cavity is stable. Along the

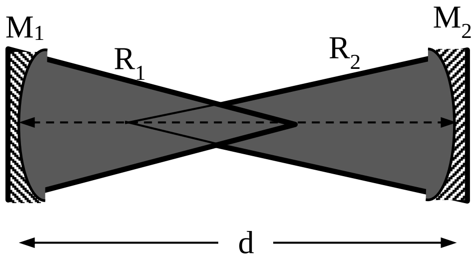


Fig.c

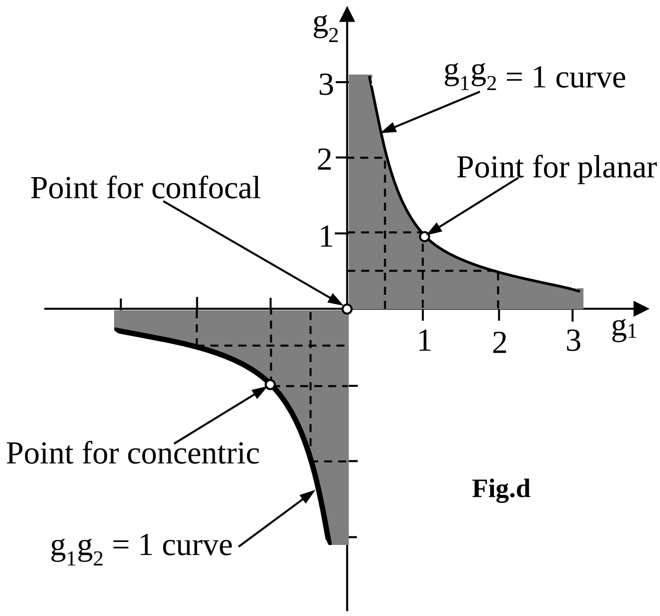


Fig.d

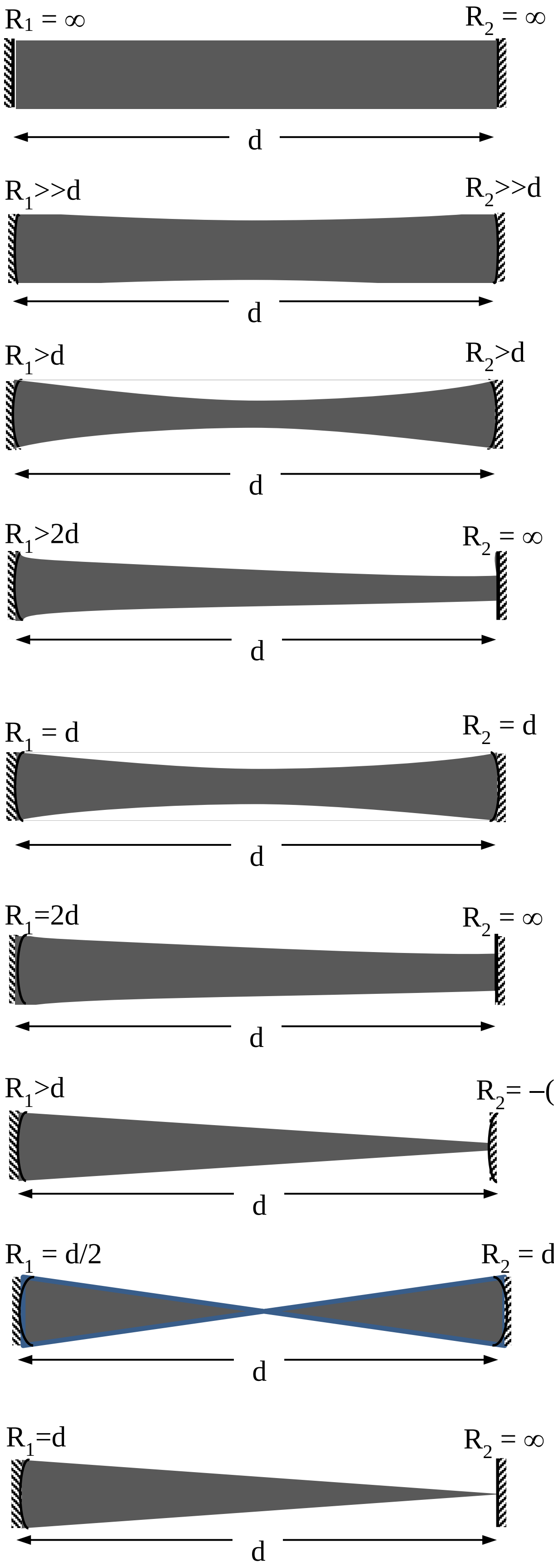
curved line $g_1g_2 = 1$. Three particular points (white spots) in the fig.b are of special interest. They are the cases of (1) two parallel plane mirrors separated by a distance d. In this case, $g_1 = g_2 = 1$, (2) confocal mirrors with $R_1 = R_2 = d$, so that $g_1 = g_2 = 0$ and (3) symmetric concentric case with $R_1 = R_2 = d/2$ so that $g_1 = g_2 = -1$. All these three points are on the edge of the stability diagram and can become highly lossy. It is wise to choose the values of R_1, R_2 and d such that the parameters g_1 and g_2 lie in the stable zone of the stability diagram.

1.13 Stable and unstable resonators

An open resonator with plane mirrors would have significant diffraction losses on account of the finite transverse size of the mirrors. These losses can be much reduced by replacing plane mirrors by spherical mirrors that provide focussing of light beam.

A spherical mirror resonator is formed by a pair of spherical mirrors (convex or concave) or a plane mirror and a concave mirror. Figure given in the next page shows the various spherical mirror resonators.

(a) Using two plane parallel mirrors separated by a distance d.
(b) Using two long radius concave mirrors, $R_1 = R_2 >> d$, facing each other separated by a distance d.
(c) Symmetric resonator using two concave mirrors, $R_1 = R_2 > d$, facing each other with separation d.
(d) Half symmetric resonator using a concave mirror with $R_1 > 2d$ and a plane mirror with a separation d.
(e) The symmetric confocal resonator using a pair of identical concave mirrors each having a radius of curvature R and with the separation between the mirrors d = R so that the foci of the mirrors coincide at the centre of the resonator.
(f) Half symmetric confocal resonator consists of a concave mirror with $R_1 = 2d$ and a plane mirror separated by a distance d.
(g) A concave-convex resonator consists of a concave mirror with $R_1 > d$ and a convex mirror with $R_2 = -(R_1 - d)$ with separation d.
(h) near concentric spherical resonator is formed by two identical concave mirrors each with a radius of curvature equal to half the distance between them. That is, $R_1 = R_2 > d/2$.



(a) Plane-parallel mirrors

(b) Long radius mirrors; $R_1 \approx R_2 >> d$

(c) Symmetric; $R_1 = R_2 > d$

(d) Half-Symmetric; $R_1 > 2d$; $R_2 = \infty$

(e) Symmetric confocal; $R_1 = R_2 = d$

(f) Half-Symmetric confocal; $R_1 = 2d$; $R_2 = \infty$

(g) Concave-convex mirrors; $R_1 > 2d$; $R_2 = -(R_1 - d)$

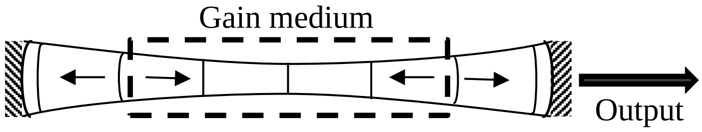
(h) Near concentric concave mirrors; $R_1 = d/2$; $R_2 = d/2$

(i) Hemispherical; $R_1 = 2d$; $R_2 = \infty$

(i) A hemispherical resonator consists of a concave mirror of R_1 = d and a plane mirror with separation d.

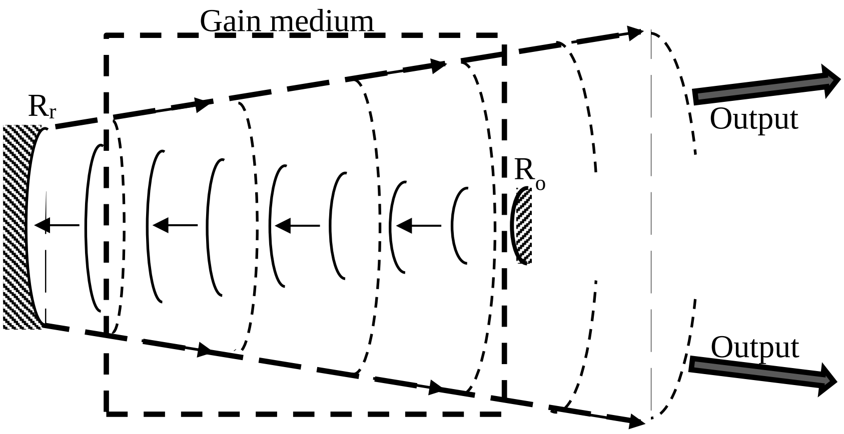
In general, one can form a spherical mirror resonator with plane, concave or convex mirrors. Depending on the curvature of the mirrors and the separation between the mirrors, the resonator is stable or unstable. In the language of geometrical optics, we can define stable and unstable resonators as follows.

Stable resonator: If a family of light rays may keep bouncing back and forth between the mirrors of the cavity indefinitely without ever escaping from the cavity, the resonator is called a stable resonator. Because of the focussing action of the mirrors the beam remains concentrated within the cavity and hence there is no loss of energy in the case of stable resonator. They satisfy the condition for stability discussed in sec.1.12.



Stable resonator

Unstable resonator: If the rays diverge away from the axis after every pass and thus escape from the resonator after a few reversals, the resonator is known as unstable resonator. Thus, for an unstable resonator there are no ray families that can bounce back and forth without escaping from the cavity. Such resonators do not satisfy the condition for stability. Unstable resonators provide useful laser output with reasonable beam quality.



Unstable resonator

The most common unstable resonator cavity has two mirrors of different diameters (different areas) and different radii of curvature. The large diameter rear mirror has a radius of curvature R_r and small diameter mirror at the output end of the cavity has a radius of curvature R_o as shown in the figure. Then,

$$\text{Unstable resonator magnification ratio, } M_{us} = \frac{R_r}{R_o}$$

Such unstable resonator cavities have transverse mode envelops. The losses in such resonators are dominated by diffraction.

The constraints associated with the unstable resonators are that $g_1 g_2 \geq 1$, or, $g_1 g_2 \leq 0$, where, $g_1 = 1 - \frac{d}{R_1} = 1 - \frac{d}{R_o}$ and $g_2 = 1 - \frac{d}{R_2} = 1 - \frac{d}{R_r}$. The unstable resonators can be classified as being either positive or negative branch according to whether,

$g_1 g_2 \geq 1$ (positive branch)

Or, $g_1 g_2 \leq 0$ (negative branch)

Figures a and b show the two types of unstable resonators that produce a collimated output beam. These are referred to as positive branch and negative branch confocal unstable resonators. The positive branch confocal unstable resonator has a constraint that $R_r - R_o = 2d$, whereas, for negative branch requires that $R_r + R_o = 2d$. Q-switched lasers and mode locked lasers are examples of unstable resonators.

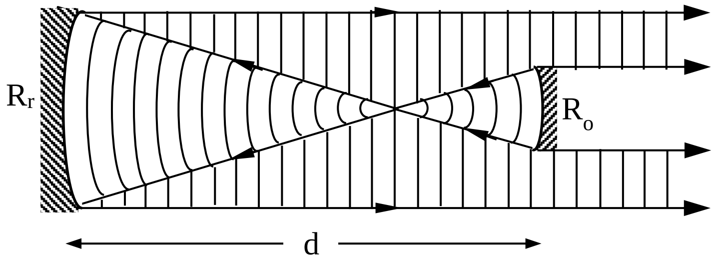


Fig.a : Negative branch confocal
$R_r + R_o = 2d$

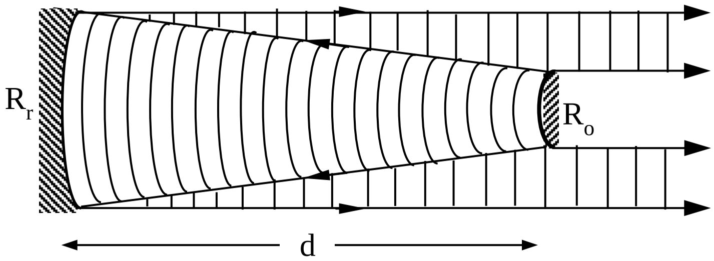


Fig.b : Positive branch confocal
$R_r - R_o = 2d$