## Chapter-5

# Optical Fibers

## 5.1 Introduction

Communication may be broadly defined as the transformation of information from one point to another. From the very beginning of the human kind people used many communication methods. In the prehistoric era fires, beacons, smoke signals, communication drums, horns, etc. were used for communication purposes. In B.C. mailing, pigeon post, etc. were used. Acoustic mechanical (sound through stretched string) telephone 1672, optical telegraphs 1790, electrical telegraphy by Samuel B Morse 1838, cable telegraph 1858, signal lamps 1867, telephones 1876 by Alexander graham bell, acoustic phonograph by Thomas Alva Edison 1877, telephony via light-beam photophones by Graham Bell 1880, wireless telegraphy Nikolai Tesla (Marcony) 1893, radio by Marconi 1896, transcontinental telephone calling by Graham Bell 1915, television 1927, radio-telephone service 1927, videophone 1930, commercial telephone service 1934 videophone network 1936, Transatlantic telephone cable 1956, satellite communication 1962 were the major events in the communication history before the introduction of fiber optic telecommunications in 1964 by Charles Kao and George Hockham. The invention of lasers in early 1960s started a leap in the history of optical communication.

For large distance communication a system, known as communication system, was required. In communication systems, the information is carried through copper wires, co-axial cables, wave guides etc. The main drawback of them is the limited band width. That is the information carrying capacity is limited. Because of the energy losses, a large number of repeaters are required. For copper cables, the repeater spacing is only a few kilometres.

## 5.2 What are optical fibers?

Since the optical frequencies (~$10^{15}$Hz) are extremely large compared with radio waves (~$10^{6}$Hz) and microwaves (~$10^{10}$Hz), light beam acting as a carrier wave is capable of carrying far more information than radio waves and microwaves. The *optical fiber* is a guiding medium, usually made of glass or plastic, through which the light waves carrying information can be transmitted efficiently. The light used is the coherent light (laser) and not the ordinary composite light. Modern optical fiber systems are able to send 140 Mbit/s information through a 220km link of one optical fiber. This is equivalent to about 450000 voice channel-km. The invention of solid state lasers and fabrication of low loss glass fibers made the optical communication easier and cheaper. The optical fibers are mainly used for long distance communication systems, LANS (Local area network systems) – a network that wires up telephones, television, computers or robots. In addition to this, they are used as sensors to detect electrical, mechanical and thermal energies, copying machines and in medical diagnostics (endoscopy).

In general, an optical fiber is a very thin and flexible medium having a cylindrical shape consisting of three sections. They are,

1. **The core:** The innermost light guiding cylindrical region having a diameter ~50μm.
2. **The cladding:** The middle region of thickness ~37.5 μm surrounded by the core. It is optically rarer than the core.
3. **The sheath:** The outermost region having a thickness ~12.5μm which protects the core and the cladding. It also gives the mechanical strength to the fiber.

In practical fibers, the cladding is usually coated with a tough resin *buffer* layer, which may be further surrounded by a *jacket* layer, usually plastic. These layers add strength to the fiber but do not contribute to its optical wave guide properties.

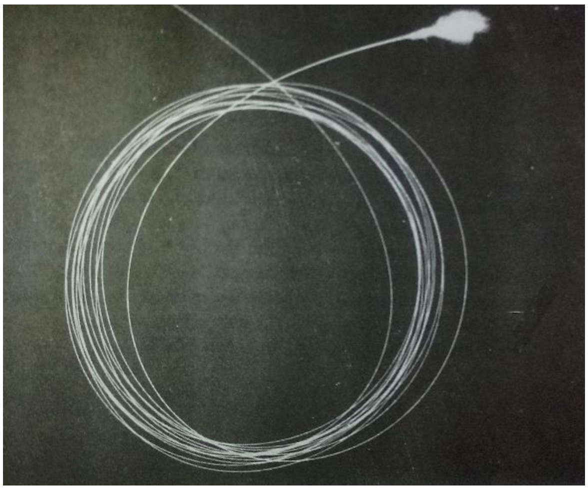


**Fig.3.1: An optical fiber carrying a light beam**

The glass fibers are drawn from a furnace containing molten silica ($SiO_2$) with small amounts of additives such as germanium dioxide ($GeO_2$) to permit control of refractive index. Most of the cable bulk is made up of strengthening and buffering materials for mechanical, moisture and chemical protection.

Usually a cable contains a large number of fibers in a single jacket. The main function of the optical fiber is to accept maximum light carrying information and transmit it with minimum loss. The light gathering power of a fiber depends on the acceptance angle.

**Optical fibers in communication system:** An optical fiber communication system has the same basic principle of any type of communication system. A general communication system consists of an information source from which the signals are carried over through a transmission medium to the destination. In an optical fiber communication system the optical fiber cable is used as the transmission medium.

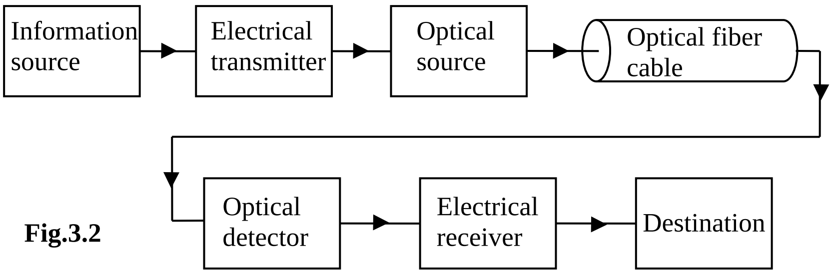



**Fig.3.2**

The figure above represents the block diagram of a typical optical fiber communication system. The information source provides electrical signals to a transmitter comprising an electrical stage which drives an optical source to give modulation of the light wave carrier. The optical source which provides the electrical-optical conversion may be either a semiconductor laser or light emitting diode (LED). The transmission medium consists of an optical fiber cable. The receiver consists of an optical detector which drives a further electrical stage. The optical detector provides the demodulation of the optical carrier. Photodiodes, in some instances phototransistors and photoconductors are utilized for the detection of the optical signal and the optical-electrical conversion. Finally, the signal from the electrical receiver is sent to the destination.

## 5.3 Importance of optical fibers

The importance of the optical fiber communication is that it has very clear cut advantages over wire or radio system. So telecommunication industries have used the fiber optic systems significantly. The optical fiber communication system is widely used in the public network applications like (1) trunk network, (2) junction network, (3) local access network, (4) submerged systems and (5) synchronous networks. The following are its main advantages.

**Advantages of fiber optic communication**

1. Very large information transmission capacity. Its data rate is much higher.
2. Very large repeater spacing because the *attenuation* is markedly lower than that of a twisted pair or coaxial cables since low loss material medium is used in optical fibers.

3. Since the optical fibers are composed of dielectric materials, they are totally isolated from extraneous interfering electromagnetic signals.
4. There is virtually no signal leakage from optical fiber. Hence the cross-talks between neighboring fibers are almost absent. Thus the transmission is more secure and private. Since the optical fibers are resistant to intrusion, they are best suited in defence communication networks.
5. Due to non-inductive non-conductive nature of the fiber they do not affect other circuits and systems by radiation and interference.
6. Since the optical fibers are immune to electromagnetic signals and also do not pick up line currents, they can be safely used in high voltage environments.
7. They are much smaller in size than other signal transmission devices.
8. Low cost of production. The basic raw material used in the fabrication of optical fiber is silica, which is abundantly available in nature.
9. Small size and lighter weight. The optical fibers are considerably thinner than coaxial or bundled twisted pair cables.
10. The system is highly reliable and maintenance is very easy. It can withstand environmental conditions, such as pollution, radiation, corrosion due to salt. Moreover, it is only nominally affected by the nuclear radiation. Its life is longer than that of copper wire.
11. No physical electrical connection is required between the sender and the receiver.
12. Bandwidth of the optical fiber is higher than that of an equivalent wire transmission line.
13. As the fibers are very good dielectrics no isolation coating is required.

## 5.4 Structure of the optical fiber and propagation of light waves in an optical fiber

The optical fiber consists of a very thin cylindrical central core made of glass or plastic. The core is cladded by a material of slightly lower refractive index. The basic principle of propagation of modulated optical signals through the fiber is the *total internal reflection*. The conditions for the light ray to undergo total internal reflection and propagates through the fiber are that it must travel from a denser to a rarer medium, such that the angle of incidence is greater than the critical angle given by $\theta_c = \mathrm{Sin}^{-1}\left(\frac{n_2}{n_1}\right)$, where $n_1$ and $n_2$ are the refractive indices of core and cladding respectively . [When refraction takes place at the interface of two media, we have the Snell's law of refraction as, $n_1 \sin\theta_1 = n_2 \sin\theta_2$. For total internal reflection $\theta_1 = \theta_c$ and $\theta_2 = 90°$.] Light rays incident on the core-cladding interface at an angle greater than the critical angle are trapped inside the core of the optical fiber. Rays making larger angles with

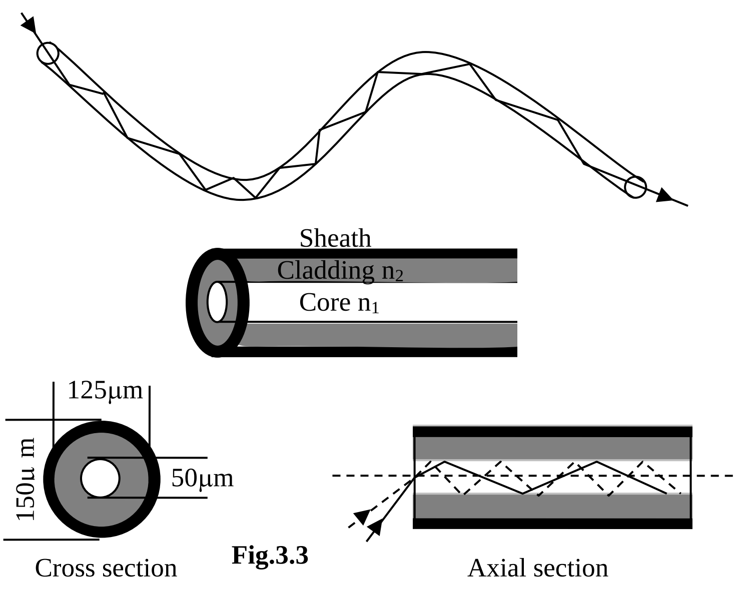


**Fig.3.3**

the axis undergo greater number of repeated reflections and hence travel greater distances and take more time to traverse the length of the fiber.

If the light that enters in the fiber at one end in proper conditions reach at the other end without considerable loss, the fiber is called a *light-guide* or sometimes *light-pipe*.

## 5.5 Acceptance angle and acceptance cone of a fiber

Consider an optical fiber with a core of material with refractive index $n_1$ (~1.48) and a cladding of refractive index $n_2$ (~1.46). Let $n_0$ be the refractive index of the medium from which the light rays enter into the fiber. For most practical purposes $n_0$ is unity. The incident ray makes an angle 'i' with the axis of the fiber. Let $\phi$ be the angle between the refracted ray and the axis. By Snell's law we have,

$$\frac{\text{Sin } i}{\text{Sin } \phi} = \frac{n_1}{n_0} \qquad (1)$$

As 'i' increases, $\phi$ increases and $\theta$ decreases. If $\theta < \theta_c$ no total internal reflection takes place and the ray escapes through the cladding. The largest value of 'i' corresponds to $\theta = \theta_c$. From the figure, $\phi = 90 - \theta$, then,

$$\text{Sin } \phi = \text{Sin}(90-\theta) = \cos\theta$$

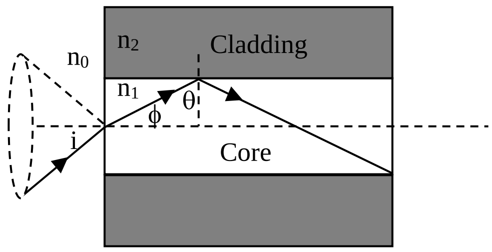



**Fig.3.4**

Then, eqn.1 becomes,

$$\text{Sin } i = \frac{n_1}{n_0} \times \cos\theta$$

And, $$\text{Sin }(i_{max}) = (\text{Sin } i)_{max} = \frac{n_1}{n_0} \cos\theta_c$$

For critical angle $\theta_c$, the Snell's law becomes,

$$\text{Sin}\theta_c = \frac{n_2}{n_1}$$

$$\therefore \quad \cos\theta_c = \sqrt{1 - \text{Sin}^2\theta_c} = \sqrt{1 - \frac{n_2^2}{n_1^2}}$$

Then, $$\text{Sin }(i_{max}) = \frac{n_1}{n_0} \times \sqrt{1 - \frac{n_2^2}{n_1^2}} = \sqrt{\frac{n_1^2 - n_2^2}{n_0^2}}$$

When the incident ray enters from air, $n_0 = 1$ and let $i_{max} = \theta_0$. Then,

$$\text{Sin}\theta_0 = \sqrt{n_1^2 - n_2^2}$$

Or, $$\theta_0 = \text{Sin}^{-1}\sqrt{n_1^2 - n_2^2} \qquad (2)$$

Eqn.2 is valid only when $0 < \left(n_1^2 - n_2^2\right) < 1$. For all values of $n_1^2 - n_2^2 > 1$, $\theta_0 = 90°$. The angle $i_{max} = \theta_0$ is called the **acceptance angle** of the fiber, which may be defined as the maximum value of the angle of incidence of the incident ray for which the ray can be guided through the fiber. The acceptance angle is a measure of the light gathering power of a fiber. Indeed, the light rays contained within a cone having semi-vertex angle $\theta_0$ are accepted and transmitted through the fiber. This cone is called the **acceptance cone** of the fiber.

## 5.6 Numerical aperture

The sine of the acceptance angle is called the numerical aperture. It is sometimes called the figure of merit for optical fiber.

i.e. Numerical aperture, N. A $= \sin\theta_0 = \sqrt{n_1^2 - n_2^2} = \sqrt{(n_1+n_2)(n_1-n_2)}$

But $n_1$ is not far different from $n_2$. Thus, $n_1 + n_2 \approx 2n_1$

Then, $$N.A \approx \sqrt{2n_1(n_1-n_2)} \approx \sqrt{2n_1^2\left(\frac{n_1-n_2}{n_1}\right)} \approx n_1\sqrt{2\Delta}$$

where, $\Delta = \frac{n_1-n_2}{n_1}$ is the fractional difference of the refractive indices of the core and the cladding. It is clear that N.A depends only on the refractive indices $n_1$ and $n_2$ of the core and the cladding respectively. Larger the value of N A, the more light will be accepted from the source by the fiber.

It has been observed that the numerical apertures for the fibers used in short distance communication are in the range of 0.4 to 0.5, whereas for long distance communication they are in the range 0.1 to 0.3. It also has been observed that smaller the numerical aperture harder to launch power into the fiber.

## 5.7 Dispersion*

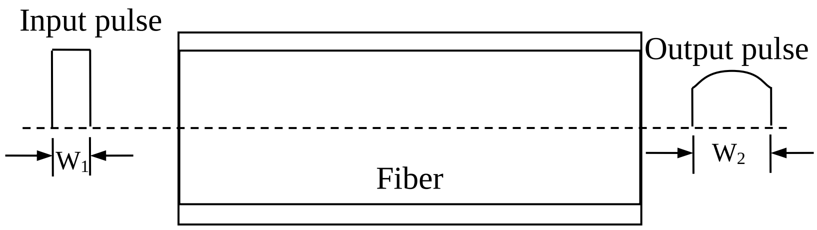



**Fig.3.5**

In a digital communication system, the information to be sent is first coded in the form of pulses of light from the transmitter. These pulses are then decoded by the receivers. A pulse of light sent into a fiber broadens in time as it propagates through the fiber. This phenomenon is known as pulse dispersion (modal dispersion), which occurs because of different times taken by the waves propagating in different directions through the fiber. The dispersion may, therefore, be defined as the output light pulse width produced by an input pulse of zero line width.

Let $W_1$ be the input pulse width and $W_2$ be the output pulse width with $W_2 > W_1$. Then the fiber dispersion dT is defined as,

$$dT = \sqrt{W_2^2 - W_1^2}$$

Dispersion is measured in units of time either in nanoseconds or picoseconds. Since the total dispersion produced by a fiber depends directly on its length, the total dispersion dT is given by,

$$dT = L \times (\text{dispersion/km})$$

where, L is the length of the fiber expressed in kilometre (km).

The dispersion is divided into (1) intermodal dispersion and (2) intramodal dispersion.

**Intermodal dispersion** is due to the difference in the propagation times for the different propagation modes. It can be shown that

$$dT = \frac{L(N.A)^2}{2n_1c} = \frac{Ln_1\Delta}{c}$$

where, N.A is the numerical aperture.

**Intramodal dispersion**, also called the chromatic dispersion, is due to the fact that the light signal propagating through the fiber consists of not a single frequency but a group of frequencies. It is related to line width of the light pulse. It is expressed in terms of picoseconds per kilometre per nanometre of line width.

The smaller the pulse dispersion, the greater will be the information carrying capacity of the fiber.

## 5.8 Classification of optical fibers

Optical fibers are classified into three groups according to the way light propagates through the fiber core. They are,

1. Stepped index monomode fiber
2. Stepped index multi-mode fiber, and
3. Graded index multi-mode fiber

Fibers which support many propagation paths or transverse modes are called multi-mode fibers (MMF), while those which can only support a single mode are called single-mode fibers (SMF). Multi-mode fibers generally have a larger core diameter, and are used for short-distance communication links and for applications where high power must be transmitted. Single-mode fibers are used for most communication links longer than 550 meters (1,800 ft).

**Stepped index fibers**

As we have already seen, the simplest type of an optical fiber consists of a core of uniform refractive index $n_1$ and a cladding of uniform refractive index $n_2$. This type of fibers is referred to as step index fibers due to the step discontinuity of the refractive index profile at the core-cladding interface.

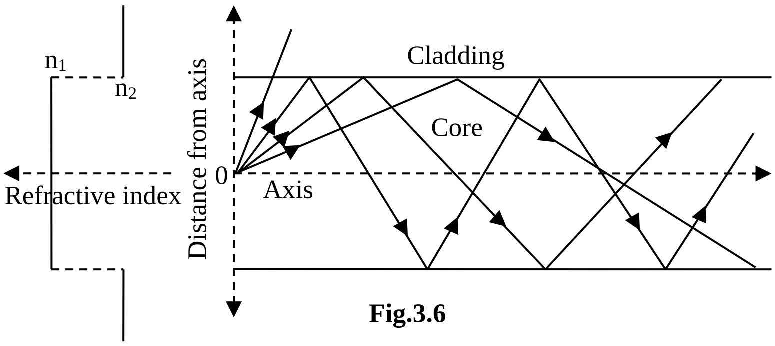



**Fig.3.6**

**Graded index fibers**

The information carrying capacity of the fiber can be improved by reducing the pulse dispersion. In an optical fiber this is achieved by using graded index fibers.

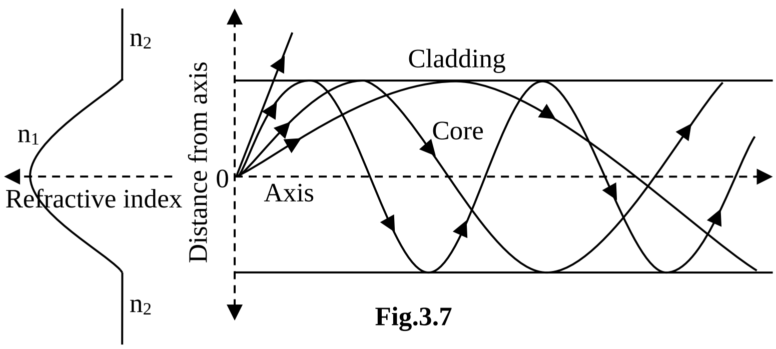



**Fig.3.7**

In a graded index fiber the refractive index of the core is not a constant but it decreases continuously in a nearly parabolic manner from maximum value at the centre of the core (axis of the fiber) to a constant value at the core-cladding interface. Since the refractive index of the core decreases as one move away from the axis, the rays entering into the fiber are continuously bent towards the axis of the fiber and the ray paths inside the fiber are sinusoidal. From the figure it is clear that light waves with large angles of incidence travel more distance than those with smaller angles. But the decrease in refractive index allows higher velocity of wave propagation. Hence all the waves travel without pulse dispersion and reach a point at the same time.

**Modes of propagation - Single mode and multimode fibers**

In the case of step index fibers we can consider the wave propagation as many rays bouncing back and forth at the core-cladding interface, since the diameter of the fiber is small or if the difference of the refractive indices of the core and cladding is small. Then one has to apply the wave theory rather than the geometrical optics. In an optical fiber there are axial rays and zigzag rays (rays other than the axial rays). Along certain paths, the zigzag rays may interfere constructively and hence the intensity will be increased. But certain others may interfere destructively and intensity may be reduced to zero. Only those paths (or directions) along which constructive interference takes place are useful for optical fiber transmission. Those particular paths or particular directions $\theta$ are called **modes**. The possible number of modes depends on the ratio $d/\lambda$ between the diameter of the core and the wavelength of the light transmitted.

The number of modes supported by a fiber is determined by a dimensionless parameter called cut-off parameter and is denoted by V. It is also known as normalized frequency cut-off or sometimes called the V number or value of the fiber. It can be shown that

$$V = \frac{2\pi}{\lambda} a(N\,A) = \frac{2\pi}{\lambda} a n_1 \sqrt{2\Delta} \qquad (1)$$

where, 'a' is the core radius, '$n_1$' is the core refractive index, $\Delta$ is the relative refractive index difference, N A is the numerical aperture and $\lambda$ is the operating wavelength.

The total number of guided modes also known as mode volume denoted by $M_s$ is related to the normalized frequency V.

For a step index fiber $$M_s \approx \frac{V^2}{2} = \frac{2\pi^2 a^2}{\lambda^2}(N\,A)^2 \qquad (3)$$

For a graded index fiber $$M_s \approx \left(\frac{p}{p+2}\right)\frac{V^2}{2} = \left(\frac{p}{p+2}\right)\frac{2\pi^2 a^2}{\lambda^2}(N\,A)^2 \qquad (4)$$

where, p is the index profile parameter. $p = \infty$ for step index profile, $p = 1$ for triangular profile and $p = 2$ for parabolic profile. The value of mode volume is doubled to account for the possible polarizations.

In a single mode fiber there is only one guided mode (only one path or direction) possible. Since there is only one ray path possible in a single mode fiber there will be no pulse dispersion.

Theory of wave guides shows that for single mode propagation in step index fibers the V value ranges from 0 to 2.405, so that the cut off value $V_c = 2.405$. From the rearrangement of eqn.1 gives the theoretical cut off wavelength for single mode operation,

$$\lambda_c = \frac{2\pi}{V_c} a(N\,A) = \frac{2\pi}{V_c} a n_1 \sqrt{2\Delta}$$

For graded index fibers the cut of value of the normalized frequency $V_c$ to support single mode propagation is given by,

$$V_c = 2.405\sqrt{\left(1+\frac{2}{p}\right)}$$

For parabolic profile $V_c = 2.405\sqrt{2}$ . In multimode fibers, a number of modes are possible.

**Comparison between single mode and multimode fibers**

| | Single mode fiber (SMF) | Multimode fiber (MMF) |
|---|---|---|
| 1 | Only a single mode propagates in SMF | Many modes propagate MMF |
| 2 | Diameter is much less than MMF | Diameter is much larger than SMF |
| 3 | Largest transmission bandwidth | Transmission bandwidth is lower |
| 4 | Exhibits lowest loss | Comparatively more loss |
| 5 | SMFs have superior transmission quality due to the absence of modal noise | Lesser transmission quality than SMF |
| 6 | Offers a substantial upgrade capability for future wide bandwidth services | MMF is not much future proof. |

**Comparison between step index and graded index fibers**

| | Step index fiber (SIF) | Graded index fiber (GIF) |
|---|---|---|
| 1 | Bandwidth is 50 MHz | Bandwidth is 200, 600, MHz etc. up to infinity theoretically |
| 2 | Mode dispersion is higher | Mode dispersion is lower |
| 3 | Numerical aperture is 0.2 to 0.5 (for 12db/km loss) | Numerical aperture is 0.16 to 0.2 (for 5 to 10db/km loss) |
| 4 | Attenuation in SIF is higher than GIF | Attenuation in GIF smaller than SIF |

**Plastic fibers**

Instead of glass we can use plastic to make optical fibers. There are two types, all plastic and plastic-clad fibers. For former type both core and cladding are plastic, while for latter only the cladding is plastic.

Because of the following disadvantages it is rarely used in commercial communication systems

1. It is more sensitive to abrasive damages.
2. It has poorer transmission characteristics.
3. Its loss is much higher.
4. Its bandwidth is lower.

All plastic fiber is stronger mechanically and it does not break easily. Thus it is easier to handle and is often used for short distance low bandwidth applications such as closed circuit television and demonstration systems etc.

## 5.9 Stepped index monomode fibers

The transit-time dispersion problem can be solved by making the core very thin. The diameter of the core is of the same order of wavelength of light wave to be propagated. This type of fiber is referred to as stepped index monomode fiber. The chief characteristics of it are,

1. Very small core diameter
2. Low numerical aperture
3. Low attenuation
4. Very high bandwidth

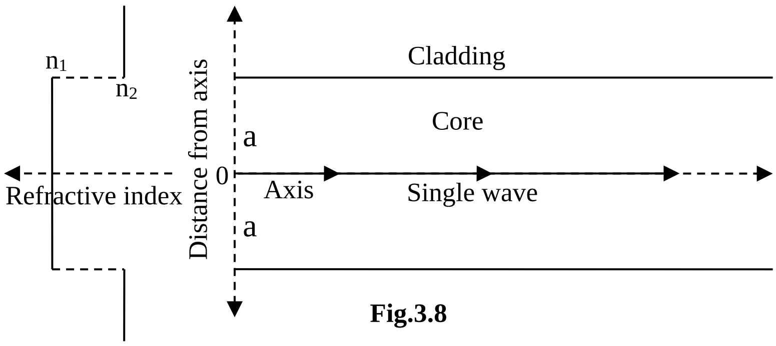



**Fig.3.8**

In order to get a single mode (with all other modes cut off) the diameter 'd' of the core must satisfy the relation,

$$d < \frac{0.766\lambda}{NA}$$

If the operating wavelength is 1.3 μm, the core diameters are in the range of 6 to 10 μm.

Figure 3.9 gives the optical power distribution across the cone of the step index single mode fiber. Another important parameter associated with the single mode fiber is the 'mode field diameter' ($2F_d$) as shown in the figure. This parameter indicates the boundary where the electric field of the optical wave falls to 1/e (=36.8%) of the field at the core centre (along the axis). It gives the light guiding property of the fiber. From the figure it is clear that a significant amount of power resides outside the core.

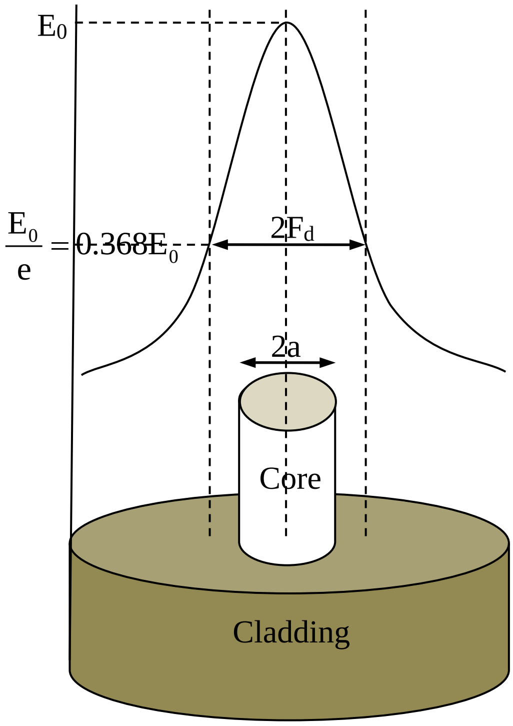


**Fig.3.9**

The distribution of the optical electric field with the radial position across the core (perpendicular to the core axis) can be described approximately by a Gaussian expression near the cut off wavelength as,

$$E(x) = E_0 e^{-\left(\frac{x}{F_d}\right)^2}$$

where, $F_d$ is half of the mode field diameter. Greater the ratio $\frac{F_d}{a}$ (mode field diameter greater than core radius), a larger amount of light propagates through the cladding. In the case of stepped index monomode fibers the cladding is very thick so that the field outside the cladding is very insignificant. The field, if any, outside the cladding will be radiated out. It can be shown that the theoretical value of mode field diameter is given by,

$$2F_d = 2a\left[0.65 + 0.434\left(\frac{\lambda}{\lambda_c}\right)^2 + 0.0149\left(\frac{\lambda}{\lambda_c}\right)^6 + \ldots\ldots\ldots\right]$$

where, $\lambda$ is the operating wavelength, $\lambda_c$ the cut off wavelength and 2a is the core diameter. From the above equation it is clear that for a given $\lambda_c$, the value of $F_d$ increases with the operating wavelength.

## 5.10 Disadvantage of monomode fiber

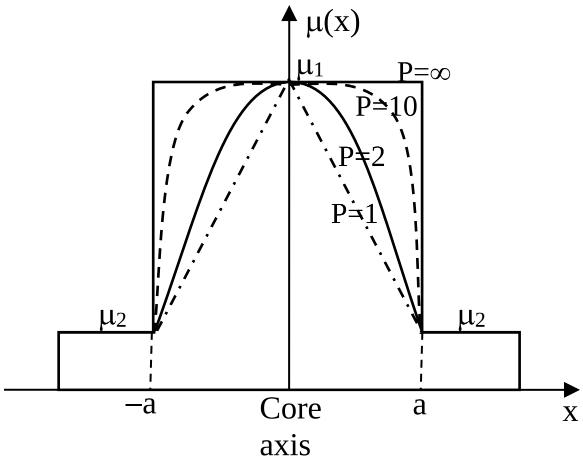
The core of the monomode fiber is very thin. So the main disadvantage of the monomode fibers is the mechanical difficulties in the manufacture, handling and splicing the fibers. Hence this type of fibers is very expensive. Monomode fiber is mainly used as undersea cables.

## 5.11 Graded index multimode fibers

Graded index multimode fibers have intermediate bandwidth and capacity. In this fiber the transit-time dispersion is avoided by a less expensive method. The refractive index of the core is gradually decreasing from the centre to the outside. In fig.3.7 the refractive index fiber together with its refractive index profile is given. From the figure it is clear that waves with larger angles of incidence should travel larger distances within the fiber. As the refractive index deceases gradually (see the parabolic variation of the profile) the optical path lengths of all the waves will be the same. Since the decrease in refractive index allows an increase in velocity the transit time of all the waves will be the same. This type of light wave propagation is referred to as graded index multimode propagation. The variation in refractive index of the core is given by,

$$\mu(x) = \begin{cases} \mu_1\left[1-2\Delta\left(\dfrac{x}{a}\right)^p\right]^{1/2} & ;\text{ for } x < a \quad (\text{core}) \\ \mu_1(1-2\Delta)^{1/2} \simeq \mu_1(1-\Delta) = \mu_2 & ;\text{ for } x \ge a \quad (\text{cladding}) \end{cases}$$

where, $\mu_1$ is the refractive index at the centre of the core, x is the distance from the centre of the core. p is the index profile, Δ is the fractional difference between the refractive indices between the core and the cladding and a is the radius of the core. The refractive index is maximum at the centre where the velocity of light is minimum and refractive index is minimum at the core boundary where the velocity of light is maximum. For an approximately parabolic refractive index profile p = 2 and the time of transit for various modes are equal. Mode volume is given by eqn.4 above.




## 5.12 Graded index monomode fibers

There are several types of single mode graded-index fibers. These fibers are not standard fibers and are typically only used in specialty applications.

## 5.13 Specification of certain optical fibers*

### (a) Multimode step index fibers

| | |
|---|---|
| Core diameter | : 50 to 400μm |
| Cladding diameter | : 125 to 500μm |
| Buffer jacket diameter | : 250 to 1000μm |
| Numerical aperture | : 0.16 to 0.50 |

Attenuation: 2.6 to 50dB per kilometer at a wavelength of 0.85μm, limited by absorption or scattering.

Fabrication: Either multicomponent glass compounds (glass-clad glass) or doped silica (silica-clad silica).

Bandwidth : 6 to 50MHz km

Applications: These fibers are best suited for short-haul, limited bandwidth and relatively low cost applications.

### (b) Multimode graded index fibers

| | |
|---|---|
| Core diameter | : 30 to 100μm |
| Cladding diameter | : 100 to 150μm |
| Buffer jacket diameter | : 250 to 1000μm |

Numerical aperture : 0.20 to 0.30

Attenuation: 2 to 10dB per kilometer at a wavelength of 0.85μm, generally limited by scattering.

Fabrication: Either multi-component glass compounds (glass-clad glass) or doped silica (silica-clad silica) with higher purity.

Bandwidth : 300MHz to 3GHz km

Applications: These fibers are best suited for medium-haul, medium to high bandwidth applications using incoherent (LEDs) and coherent (injection lasers) multimode lasers.

**(c) Single-mode fibers**: With step index or graded index profile.

Core diameter : 5 to 10μm, around 8.5μm
Cladding diameter : Generally 125μm
Buffer jacket diameter : 250 to 1000μm
Numerical aperture : 0.08 to 0.15, around 0.10

Attenuation: 2 to 5dB per kilometer at a wavelength of 0.85μm, limited by scattering.

Fabrication: Doped silica (silica-clad silica).

Bandwidth : Greater than 500MHz km

Applications: These fibers are best suited for high bandwidth very long-haul applications using single mode injection laser sources.

**(d) Plastic-clad fibers**: With step index or graded index profile.

Core diameter : Step index : 100 to 500μm
Graded index : 50 to 100μm
Cladding diameter: Step index : 300 to 800μm
Graded index : 125 to 150μm

Buffer jacket diameter:Step index: 500 to 1000μm
Graded index : 250 to 1000μm

Numerical aperture: Step index : 0.20 to 0.50
Graded index : 0.20 to 0.30

Attenuation : Step index : 5 to 50dB per kilometer.
Graded index : 4 to 15 dB per kilometer.

Fabrication: Plastic clad and glass core which is silica (PCS).

Applications: Improved performance in certain environments and slightly cheaper than glass fibers.

**(e) All-plastic fibers :** (exclusively multimode step index)

Core diameter : 200 to 600μm
Cladding diameter : 450 to 1000μm
Buffer jacket diameter : No need of buffer jacket
Numerical aperture : 0.50 to 0.60

Attenuation: 50 to 1000dB per kilometer at a wavelength of 0.65μm.

Bandwidth: Usually not specified as transmission is generally limited to tens of meters.

Applications: These fibers can only be used for very short-haul low cost links (i.e. in house). However, fiber coupling and termination are relatively easy and do not require sophisticated techniques.

## 5.14 Optical fibers as cylindrical waveguides

In an isotropic, linear, non-conducting, nonmagnetic and inhomogeneous (varying refractive index) medium with no free charges, ρ = 0, J = 0, $\mu_r = 1$, D = εE = $\varepsilon_r\varepsilon_0$E = K$\varepsilon_0$E = $n^2\varepsilon_0$E, where K = $n^2$ is the dielectric constant and n is the refractive index of the medium. The Maxwell's equation become,

$$\nabla.\mathbf{D} = \rho = 0 \tag{1}$$

$$\nabla.\mathbf{B} = 0 \tag{2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{H} = \mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} = \varepsilon_0 n^2\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

Eqn.1 can be written as,

$$\nabla.\mathbf{D} = 0$$

i.e. $$\nabla.\left(\varepsilon_0 n^2\mathbf{E}\right) = 0$$

In an inhomogeneous medium there is a spatial variation of refractive index (as in the case of a graded index optical fiber). Thus, the above equation becomes,

$$\varepsilon_0\left\{\nabla\left(n^2\right).\mathbf{E}+n^2\nabla.\mathbf{E}\right\} = 0$$

i.e. $$\nabla.\mathbf{E} = -\frac{1}{n^2}\nabla\left(n^2\right).\mathbf{E} \tag{5}$$

Taking curl of eqn.3 we obtain,

$$\nabla\times(\nabla\times\mathbf{E}) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) = -\mu_0\frac{\partial}{\partial t}\left(\varepsilon_0 n^2\frac{\partial\mathbf{E}}{\partial t}\right) = -\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{E}}{\partial t^2}$$

i.e. $$\nabla(\nabla.\mathbf{E})-(\nabla.\nabla)\mathbf{E} = -\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Using eqn.5 we get,

$$-\nabla\left(\frac{1}{n^2}\nabla n^2.\mathbf{E}\right)-\nabla^2\mathbf{E} = -\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{E}}{\partial t^2}$$

i.e. $$\nabla^2\mathbf{E}+\nabla\left(\frac{1}{n^2}\nabla n^2.\mathbf{E}\right)-\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{6}$$

Taking curl of eqn.4 we get,

$$\nabla\times(\nabla\times\mathbf{H}) = \nabla\times\left(\varepsilon_0 n^2\frac{\partial\mathbf{E}}{\partial t}\right)$$

i.e. $$\nabla(\nabla.\mathbf{H})-(\nabla.\nabla)\mathbf{H} = \varepsilon_0\frac{\partial}{\partial t}\left\{\nabla\times\left(n^2\mathbf{E}\right)\right\}$$

Using eqn.2, $$-\nabla^2\mathbf{H} = \varepsilon_0\frac{\partial}{\partial t}\left\{\nabla n^2\times\mathbf{E}+n^2\nabla\times\mathbf{E}\right\} = \varepsilon_0\left\{\nabla n^2\times\frac{\partial\mathbf{E}}{\partial t}+n^2\frac{\partial}{\partial t}(\nabla\times\mathbf{E})\right\}$$

Using eqn.4 and 3 in RHS, we get,

$$-\nabla^2\mathbf{H} = \varepsilon_0\left\{\nabla n^2\times\left(\frac{1}{\varepsilon_0 n^2}\nabla\times\mathbf{H}\right)-\mu_0 n^2\frac{\partial^2\mathbf{H}}{\partial t^2}\right\}$$

i.e. $$\nabla^2\mathbf{H}+\frac{1}{n^2}\nabla n^2\times(\nabla\times\mathbf{H})-\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \tag{7}$$

Now for an infinitely extended homogeneous medium n is constant. Then eqnns.6 and7 reduce to,

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 n^2 \frac{\partial^2\mathbf{E}}{\partial t^2} \qquad (8)$$

$$\nabla^2\mathbf{H} = \mu_0\varepsilon_0 n^2 \frac{\partial^2\mathbf{H}}{\partial t^2} \qquad (9)$$

Since the optical fiber is cylindrical in shape we use the cylindrical coordinate system for the analysis of the propagation of the electromagnetic wave through the fiber. Using cylindrical coordinate system a point is specified by the radial distance 'r' from the axis, the azimuthal angle $\phi$ and the Z-coordinate z as shown in fig.3.10. If $\beta$ is the z-component of the propagation vector the electric and magnetic fields of the electromagnetic wave propagating in the z-direction through the homogeneous medium are given by the solutions eqns.8 and 9,

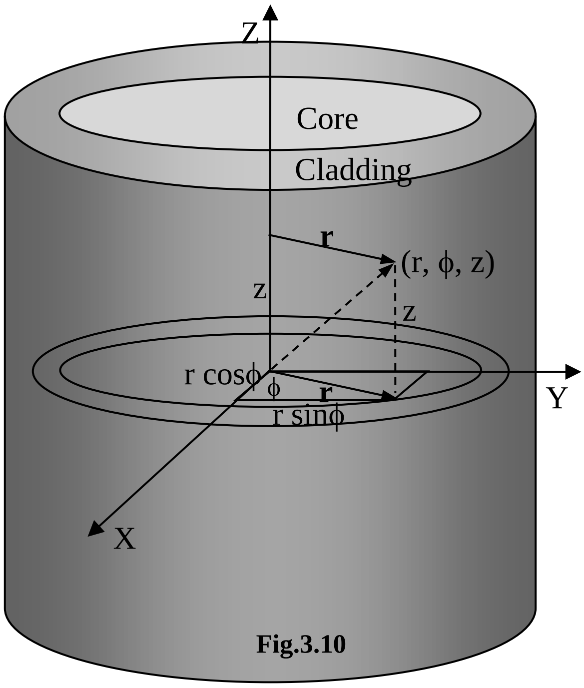


**Fig.3.10**

$$\mathbf{E} = \mathbf{E}_0(r,\phi)e^{i(\omega t-\beta z)} \qquad (10)$$

i.e. $$\hat{\mathbf{r}}E_r + \hat{\boldsymbol{\phi}}E_\phi + \hat{\mathbf{z}}E_z = \left(\hat{\mathbf{r}}E_{r0} + \hat{\boldsymbol{\phi}}E_{\phi 0} + \hat{\mathbf{z}}E_{z0}\right)e^{i(\omega t-\beta z)} \qquad (10a)$$

Thus, we get

$$E_r(r,\phi,z,t) = E_{r0}(r,\phi)e^{i(\omega t-\beta z)} \qquad (10b)$$

$$E_\phi(r,\phi,z,t) = E_{\phi 0}(r,\phi)e^{i(\omega t-\beta z)} \qquad (10c)$$

$$E_z(r,\phi,z,t) = E_{z0}(r,\phi)e^{i(\omega t-\beta z)} \qquad (10d)$$

And,

$$\mathbf{H} = \mathbf{H}_0(r,\phi)e^{i(\omega t-\beta z)} \qquad (11)$$

i.e. $$\hat{\mathbf{r}}H_r + \hat{\boldsymbol{\phi}}H_\phi + \hat{\mathbf{z}}H_z = \left(\hat{\mathbf{r}}H_{r0} + \hat{\boldsymbol{\phi}}H_{\phi 0} + \hat{\mathbf{z}}H_{z0}\right)e^{i(\omega t-\beta z)} \qquad (11a)$$

Thus, we get

$$H_r(r,\phi,z,t) = H_{r0}(r,\phi)e^{i(\omega t-\beta z)} \qquad (11b)$$

$$H_\phi(r,\phi,z,t) = H_{\phi 0}(r,\phi)e^{i(\omega t-\beta z)} \qquad (11c)$$

$$H_z(r,\phi,z,t) = H_{z0}(r,\phi)e^{i(\omega t-\beta z)} \qquad (11d)$$

In cylindrical coordinate system the curl of a vector point function **F** can be written as,

$$\nabla\times\mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\phi\right) - \frac{\partial F_r}{\partial\phi}\right]\hat{\mathbf{z}} \qquad (12)$$

Using eqn.12, Maxwell's third equation in homogeneous medium becomes,

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t}$$

i.e. $$\left(\frac{1}{r}\frac{\partial E_z}{\partial\phi}-\frac{\partial E_\phi}{\partial z}\right)\hat{\mathbf{r}}+\left(\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}\right)\hat{\boldsymbol{\phi}}+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\phi\right)-\frac{\partial E_r}{\partial\phi}\right]\hat{\mathbf{z}}=-\mu_0\frac{\partial}{\partial t}\left(\hat{\mathbf{r}}H_r+\hat{\boldsymbol{\phi}}H_\phi+\hat{\mathbf{z}}H_z\right) \quad (13)$$

Equating the coefficients of $\hat{\mathbf{r}}$ on both sides of eqn.13 we get

$$\frac{1}{r}\frac{\partial E_z}{\partial\phi}-\frac{\partial E_\phi}{\partial z} = -\mu_0\frac{\partial H_r}{\partial t}$$

Using eqn.10c and 11b,

$$\frac{1}{r}\left(\frac{\partial E_z}{\partial\phi}+i\beta rE_\phi\right) = -i\omega\mu_0 H_r \quad (13a)$$

Equating the coefficients of $\hat{\boldsymbol{\phi}}$ on both sides of eqn.13 we get,

$$\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r} = -\mu_0\frac{\partial H_\phi}{\partial t}$$

Using eqns. 10b and 11c, we get,

$$i\beta E_r+\frac{\partial E_z}{\partial r} = i\omega\mu_0 H_\phi \quad (13b)$$

Equating the coefficients of $\hat{\mathbf{z}}$ on both sides of eqn.13 we get.

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\phi\right)-\frac{\partial E_r}{\partial\phi}\right] = -\mu_0\frac{\partial H_z}{\partial t}$$

Using eqn.11d

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\phi\right)-\frac{\partial E_r}{\partial\phi}\right] = -i\omega\mu_0 H_z \quad (13c)$$

Similarly from Maxwell's fourth equation we get

$$\frac{1}{r}\left(\frac{\partial H_z}{\partial\phi}+i\beta rH_\phi\right) = i\omega\varepsilon_0 n^2 E_r \quad (14a)$$

$$i\beta H_r+\frac{\partial H_z}{\partial r} = -i\omega\varepsilon_0 n^2 E_\phi \quad (14b)$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rH_\phi\right)-\frac{\partial H_r}{\partial\phi}\right] = i\omega\varepsilon_0 n^2 E_z \quad (14c)$$

Knowing $E_z$ and $H_z$ one can solve the set of eqns.13 and 14 for other transverse components of E and H. From eqns.13a and 14b we get,

$$\omega\mu_0 H_r+\beta E_\phi = \frac{i}{r}\frac{\partial E_z}{\partial\phi} \quad (15)$$

$$\beta H_r+\omega\varepsilon_0 n^2 E_\phi = i\frac{\partial H_z}{\partial r} \quad (16)$$

Multiplying eqn.15 by $\omega\varepsilon_0 n^2$ and eqn.16 by β and subtracting,

$$\omega\varepsilon_0 n^2\omega\mu_0 H_r+\omega\varepsilon_0 n^2\beta E_\phi = \frac{i}{r}\omega\varepsilon_0 n^2\frac{\partial E_z}{\partial\phi}$$

$$\beta^2 H_r+\beta\omega\varepsilon_0 n^2 E_\phi = i\beta\frac{\partial H_z}{\partial r}$$

$$\left(\mu_0\varepsilon_0\omega^2 n^2-\beta^2\right)H_r = -i\left(\beta\frac{\partial H_z}{\partial r}-\frac{\omega\varepsilon_0 n^2}{r}\frac{\partial E_z}{\partial\phi}\right)$$

$$H_r = -\frac{i}{\left(\mu_0\varepsilon_0\omega^2 n^2 - \beta^2\right)}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\varepsilon_0 n^2}{r}\frac{\partial E_z}{\partial \phi}\right)$$

$$H_r = -\frac{i}{p^2}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\varepsilon_0 n^2}{r}\frac{\partial E_z}{\partial \phi}\right) \quad (17a)$$

where, $p^2 = \mu_0\varepsilon_0\omega^2 n^2 - \beta^2 = k^2 - \beta^2$ (18)

Multiplying eqn.16 by $\omega\mu_0$ and eqn.15 by β and subtracting,

$$\left(\mu_0\varepsilon_0\omega^2 n^2 - \beta^2\right)E_\phi = -i\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu_0\frac{\partial H_z}{\partial r}\right)$$

$$E_\phi = -\frac{i}{p^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu_0\frac{\partial H_z}{\partial r}\right) \quad (17b)$$

Similarly, from eqns.13b and 14a we get,

$$-\beta E_r + \omega\mu_0 H_\phi = -i\frac{\partial E_z}{\partial r} \quad (18)$$

$$\omega\varepsilon_0 n^2 E_r - \beta H_\phi = -\frac{i}{r}\frac{\partial H_z}{\partial \phi} \quad (19)$$

Multiplying eqn.18 by β and 19 by $\omega\mu_0$ and adding

$$\left(\mu_0\varepsilon_0\omega^2 n^2 - \beta^2\right)E_r = -i\left(\beta\frac{\partial E_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_r = -\frac{i}{p^2}\left(\beta\frac{\partial E_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial H_z}{\partial \phi}\right) \quad (17c)$$

Multiplying eqn.18 by $\omega\varepsilon_0 n^2$ and eqn.19 by β and adding, we get,

$$\left(\mu_0\varepsilon_0\omega^2 n^2 - \beta^2\right)H_\phi = -i\left(\frac{\beta}{r}\frac{\partial H_z}{\partial \phi} + \omega\varepsilon_0 n^2\frac{\partial E_z}{\partial r}\right)$$

$$H_\phi = -\frac{i}{p^2}\left(\frac{\beta}{r}\frac{\partial H_z}{\partial \phi} + \omega\varepsilon_0 n^2\frac{\partial E_z}{\partial r}\right) \quad (17d)$$

[Recall that $\varepsilon_0 n^2 = \varepsilon_0 K = \varepsilon_0\varepsilon_r = \varepsilon$ and for magnetic medium instead of $\mu_0$ we must use μ in all equations, from eqn.8 onwards].

Substituting for $H_r$ (eqn.17a) and $H_\phi$ (eqn.17.d) in eqn.14c we get,

$$\frac{i}{rp^2}\left[-\frac{\partial}{\partial r}\left(\beta\frac{\partial H_z}{\partial \phi} + \omega\varepsilon_0 n^2 r\frac{\partial E_z}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\varepsilon_0 n^2}{r}\frac{\partial E_z}{\partial \phi}\right)\right] = i\omega\varepsilon_0 n^2 E_z$$

$$\frac{1}{rp^2}\left[-\beta\frac{\partial}{\partial r}\frac{\partial H_z}{\partial \phi} - \omega\varepsilon_0 n^2\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \beta\frac{\partial}{\partial \phi}\frac{\partial H_z}{\partial r} - \frac{\omega\varepsilon_0 n^2}{r}\frac{\partial}{\partial \phi}\left(\frac{\partial E_z}{\partial \phi}\right)\right] = i\omega\varepsilon_0 n^2 E_z$$

$$-\frac{1}{r}\frac{\partial E_z}{\partial r} - \frac{\partial^2 E_z}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} = p^2 E_z$$

i.e.

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + p^2 E_z = 0 \quad (20)$$

Similarly substituting for $E_r$ (eqn.17c) and $E_\phi$ (eqn.17.b) in eqn.13.c, we get,

$$\frac{\partial^2 H_z}{\partial r^2}+\frac{1}{r}\frac{\partial H_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2}+p^2 H_z = 0 \qquad (21)$$

It is to be noted that eqn.20 contains only $E_z$ whereas eqn.21 contains only $H_z$. This shows that the longitudinal components of **E** and **H** are uncoupled and can be chosen arbitrarily, provided that they satisfy the eqns.20 and 21. However, in general, coupling of $E_z$ and $H_z$ is required by the boundary conditions of the electromagnetic field components. If the boundary conditions do not lead to coupling between field components, mode solutions can be obtained in which either $E_z = 0$ or $H_z = 0$. When $E_z = 0$, the modes are called transverse electric (TE) modes and when $H_z = 0$ they are called transverse magnetic (TM) modes. Hybrid modes exist if both $E_z$ and $H_z$ are nonzero. These are designated as HE modes if $H_z > E_z$ or EH modes if $E_z > H_z$. The hybrid modes present in the waveguides makes their analysis more complex than the simpler case of hollow metallic waveguides where only TE and TM modes are found.

## 5.15 Scalar wave equation and the modes of a fiber

For an inhomogeneous medium, since the refractive index is not a constant, the electric field **E** and the magnetic field **H** satisfy the equations, (eqns.6 and 7 sec.3.14)

$$\nabla^2\mathbf{E}+\nabla\left(\frac{1}{n^2}\nabla n^2.\mathbf{E}\right)-\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \qquad (1)$$

$$\nabla^2\mathbf{H}+\frac{1}{n^2}\nabla n^2\times(\nabla\times\mathbf{H})-\mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \qquad (2)$$

In an infinitely extended homogeneous medium, since n is constant, the second term on the LHSs of the above equations are zero everywhere. Then the above equations become,

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{E}}{\partial t^2}$$

i.e. $$\nabla^2\left(\hat{\mathbf{i}}E_x+\hat{\mathbf{j}}E_y+\hat{\mathbf{k}}E_z\right) = \mu_0\varepsilon_0 n^2\frac{\partial^2}{\partial t^2}\left(\hat{\mathbf{i}}E_x+\hat{\mathbf{j}}E_y+\hat{\mathbf{k}}E_z\right) \qquad (3a)$$

Or, $$\nabla^2\left(\mathbf{E}_x+\mathbf{E}_y+\mathbf{E}_z\right) = \mu_0\varepsilon_0 n^2\frac{\partial^2}{\partial t^2}\left(\mathbf{E}_x+\mathbf{E}_y+\mathbf{E}_z\right) \qquad (3b)$$

And, $$\nabla^2\mathbf{H} = \mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{H}}{\partial t^2} \qquad (4a)$$

i.e. $$\nabla^2\left(\mathbf{H}_x+\mathbf{H}_y+\mathbf{H}_z\right) = \mu_0\varepsilon_0 n^2\frac{\partial^2}{\partial t^2}\left(\mathbf{H}_x+\mathbf{H}_y+\mathbf{H}_z\right) \qquad (4b)$$

Now we represent one of the Cartesian components of eqns.3b by $\mathbf{\Psi}$. Then by eqn.3b we see that each Cartesian component of the electric field satisfies the wave equation,

$$\nabla^2\mathbf{\Psi} = \mu_0\varepsilon_0 n^2\frac{\partial^2\mathbf{\Psi}}{\partial t^2} \qquad (5)$$

The solutions of eqn.5 can be written in the form of plane waves, $\mathbf{\Psi}=\mathbf{\Psi}_0 e^{-i(\omega t-\mathbf{k.r})}$. We can also easily show the waves are transverse.

For a medium having weak inhomogeneity, i.e. the variation of n is small in a region of the order of λ, the second term of the LHS of eqn.1 can be assumed to be negligible and hence the waves can be assumed to be nearly transverse with the transverse component of the electric field satisfies the eqn.5. This equation is obtained after neglecting the term on $\nabla n^2$ in the eqn.1. This is known as *scalar wave approximation* and in this approximation the modes have been assumed to be nearly transverse and can have arbitrary state of polarization. Thus with respect to the given Cartesian coordinate system the two independent sets of modes can be assumed to be the x-polarized and the y-polarized. In the scalar approximation these polarizations can have the same propagation constants. These linearly polarized modes are usually referred to as LP (linearly polarized) modes. When $n_1 \approx n_2$ the modes are nearly transverse and the propagation constants of these modes, TE (transverse electric) and TM (transverse magnetic) modes, are almost equal.

Since $n^2$ depends only on the transverse coordinates r and $\phi$ we may write,

$$\boldsymbol{\Psi}(r, \phi, z, t) = \psi(r, \phi)e^{i(\omega t-\beta z)} \tag{6}$$

where, $\omega$ is the angular frequency and $\beta$ is the propagation constant. Eqn.6 represents the modes of the system. Using eqn.6 in eqn.5 we get,

$$\boldsymbol{\nabla}^2\left\{\psi(r, \phi)e^{i(\omega t-\beta z)}\right\}-\mu_0\varepsilon_0 n^2(r, \phi)\frac{\partial^2}{\partial t^2}\left\{\psi(r, \phi)e^{i(\omega t-\beta z)}\right\} = 0$$

i.e. $$\boldsymbol{\nabla}^2\left\{\psi(r, \phi)\right\}e^{i(\omega t-\beta z)}+\psi(r, \phi)\,\boldsymbol{\nabla}^2\left\{e^{i(\omega t-\beta z)}\right\}-\mu_0\varepsilon_0 n^2(r, \phi)\psi(r, \phi)\frac{\partial^2}{\partial t^2}\left\{e^{i(\omega t-\beta z)}\right\} = 0$$

i.e. $$\boldsymbol{\nabla}^2\left\{\psi(r, \phi)\right\}e^{i(\omega t-\beta z)}-\beta^2\psi(r, \phi)e^{i(\omega t-\beta z)}+\mu_0\varepsilon_0 n^2(r, \phi)\omega^2\psi(r, \phi)e^{i(\omega t-\beta z)} = 0$$

i.e. $$\boldsymbol{\nabla}^2\left\{\psi(r, \phi)\right\}+\left[\frac{\omega^2}{c^2}n^2(r, \phi)-\beta^2\right]\psi(r, \phi) = 0 \tag{7}$$

If n varies with r only (i.e. the case of optical fibers), owing to the cylindrical symmetry of the situation we can use the cylindrical coordinate system, in which we have,

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2} = \frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}$$

$$\boldsymbol{\nabla}^2\left\{\psi(r, \phi)\right\} = \frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial \phi^2} \tag{8}$$

since, $\psi(r, \phi)$ is independent of z, $\frac{\partial^2\psi}{\partial z^2} = 0$.

Then using eqn.8 in eqn.7 we obtain,

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}+\left[\frac{\omega^2}{c^2}n^2(r)-\beta^2\right]\psi = 0$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}+\left[k_0^2 n^2(r)-\beta^2\right]\psi = 0 \tag{9}$$

where, $k_0 = \frac{\omega}{c} = \frac{2\pi\nu_0}{c} = \frac{2\pi}{\lambda_0}$ is the free space propagation constant. Since $\psi$ is a function of r and $\phi$ we use the method of separation of variables to solve the eqn.9. That is,

$$\psi(r, \phi) = R(r)\Phi(\phi) \tag{10}$$

Then eqn.9 becomes

$$\frac{\partial^2}{\partial r^2}\left\{R(r)\Phi(\phi)\right\}+\frac{1}{r}\frac{\partial}{\partial r}\left\{R(r)\Phi(\phi)\right\}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\left\{R(r)\Phi(\phi)\right\}+\left[k_0^2 n^2(r)-\beta^2\right]\left\{R(r)\Phi(\phi)\right\} = 0$$

$$\frac{\partial^2 R(r)}{\partial r^2}\Phi(\phi)+\frac{1}{r}\frac{\partial R(r)}{\partial r}\Phi(\phi)+\frac{1}{r^2}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}R(r)+\left[k_0^2 n^2(r)-\beta^2\right]\left\{R(r)\Phi(\phi)\right\} = 0$$

Multiplying with $r^2$ and dividing with $R(r)\Phi(\phi)$ throughout and rearranging we get,

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+r^2\left[k_0^2n^2(r)-\beta^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \quad (11)$$

Since both sides of eqn.11 are equations of different variables eqn.11 is correct only if both sides are equal to the same constant, say $l^2$. Then we get two equations,

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = l^2$$

i.e.
$$\frac{d^2\Phi}{d\phi^2}+l^2\Phi = 0 \quad (12)$$

and,
$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+r^2\left[k_0^2n^2(r)-\beta^2\right] = l^2$$

i.e.
$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left[r^2\left\{k_0^2n^2(r)-\beta^2\right\}-l^2\right]R = 0 \quad (13)$$

The solution of eqn.12 can be written as,

$$\Phi = Ae^{il\phi}$$

Since $\Phi$ is single valued, $\Phi(\phi) = \Phi(\phi+2\pi)$

i.e. $Ae^{il\phi} = Ae^{il(\phi+2\pi)} = Ae^{il\phi}e^{il2\pi}$

i.e. $1 = e^{il2\pi} = \cos(2\pi l)+i\sin(2\pi l)$

This equation is true when $l$ = 0, 1, 2, 3, ......... The negative integers are avoided since they correspond to the same field distribution. Later we will see that modes with $l \geq 1$ are four fold degenerate and mode with $l = 0$ is twofold degenerate.

Before going to the solution of eqn.13 we make some general comments on the solutions of it for an arbitrary cylindrically symmetric index profile of the core. The refractive index of the core at its axis (r = 0) is $n_1$. It decreases monotonically from the axis to a value $n_2$ at the core-cladding interface at r = a. The solutions of eqn.13 can be divided into two distinct classes.

(a) $k_0^2n_1^2 > \beta^2 > k_0^2n_2^2$

For $\beta^2$ lying in the above range the solutions (fields) R(r) are oscillatory in the core and gets decayed in the cladding. It is very important that $\beta^2$ cannot have all values but it can have only certain discrete values known as the *guided modes* of the system. For a given value of $l$ (mentioned in the solution of eqn.12) there are several guided modes designated as $LP_{lm}$ modes with m = 1, 2, 3, ......... Here LP stands for linearly polarized. Further these modes are the solutions of scalar wave equations (eqn.5) they can be assumed to satisfy the orthonormality condition given by,

$$\int_0^\infty\int_0^{2\pi}\psi_{lm}^*(r,\phi)\,\psi_{l'm'}(r,\phi)\,r\,dr\,d\phi = \delta_{ll'}\delta_{mm'} \quad (14)$$

[Remember that the area element is dA = dr.rd$\phi$]
We can write,

$$E_r = E_x\cos\phi+E_y\sin\phi \quad (15a)$$

$$E_\phi = -E_x\sin\phi+E_y\cos\phi \quad (15b)$$

Thus for the given ($l$, m) the four degenerate $LP_{lm}$ modes are

$$\mathbf{E}_1 = \hat{\mathbf{x}}\psi_{lm}\cos l\phi \quad (16a)$$

$$\mathbf{E}_2 = \hat{\mathbf{y}}\psi_{lm}\cos l\phi \quad (16b)$$

$$\mathbf{E}_3 = \hat{\mathbf{x}}\psi_{lm} \sin l\phi \quad (16c)$$

$$\mathbf{E}_4 = \hat{\mathbf{y}}\psi_{lm} \sin l\phi \quad (16d)$$

(b) $\beta^2 < k_0^2 n_2^2$

For such β values, the fields are oscillatory even in the cladding and β can have continuum values. These are known as the *radiation modes*.

The guided and the radiation modes form a complete set of modes in the sense that an arbitrary field distribution can be expanded in terms of these modes. That is,

$$\psi(x, y, z) = \sum_{\nu} a_\nu \psi_\nu(x, y) e^{-i\beta_\nu z} + \int a(\beta)\psi(\beta, x, y) e^{-i\beta z} d\beta \quad (17)$$

where the first term represents a sum over discrete modes and the second term an integral over the continuum of modes. The quantity $|a_\nu|^2$ is proportional to the power carried by the $\nu^{th}$ mode. The constants $a_\nu$ can be determined by knowing the incident field at z = 0 and using the orthonormality condition.

The importance of the calculation of modal field distributions and the corresponding propagation constants are,

(a) Knowing the frequency dependence of the propagation constant one can calculate the temporal broadening of a pulse which determines the information-carrying capacity.
(b) Knowledge of the modal field distribution is essential for the calculation of the excitation efficiencies, splice losses at joints and in the development of new fiber optic devices like directional couplers etc.

## 5.16.1 Modal analysis for a step index fiber

For a step index fiber,

$$n(r) = \begin{cases} n_1 & ; \text{ for } r < a \\ n_2 & ; \text{ for } r > a \end{cases} \quad (1)$$

For most practical fibers used in communication the relative refractive index difference $\frac{n_1 - n_2}{n_1} << 1$. In such a case the radial part of the transverse component of the electric field satisfies the equation given by (eqn.13 sec 3.15),

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left[\left\{k_0^2 n^2(r) - \beta^2\right\} r^2 - l^2\right] R = 0 \quad (2)$$

The complete transverse field is given by (eqn.6 sec 3.15)

$$\mathbf{\Psi}(r, \phi, z, t) = \psi(r, \phi) e^{i(\omega t - \beta z)} \quad (3)$$

But, $\psi(r, \phi) = R(r)\Phi(\phi)$

Then, $\mathbf{\Psi}(r, \phi, z, t) = R(r)\Phi(\phi) e^{i(\omega t - \beta z)}$

Since as we have seen earlier $\Phi(\phi)$ is either a cosine function or a sine function of φ we can write,

$$\mathbf{\Psi}(r, \phi, z, t) = R(r)\Phi(\phi) e^{i(\omega t - \beta z)} = R(r)\begin{bmatrix} \cos(l\phi) \\ \sin(l\phi) \end{bmatrix} e^{i(\omega t - \beta z)} \quad (4)$$

Using eqn.1 in eqn.2, we obtain,

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left[\left(k_0^2n_1^2-\beta^2\right)r^2-l^2\right]R = 0 \quad ;\text{ for } r < a$$

i.e. $$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left[a^2\left(k_0^2n_1^2-\beta^2\right)\frac{r^2}{a^2}-l^2\right]R = 0 \quad ;\text{ for } r < a$$

i.e. $$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left[\frac{U^2r^2}{a^2}-l^2\right]R = 0 \quad ;\text{ for } r < a \qquad (5a)$$

where, $U=a\left(k_0^2n_1^2-\beta^2\right)^{1/2}$ (6a)

and $$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left[\left(k_0^2n_2^2-\beta^2\right)r^2-l^2\right]R = 0 \quad ;\text{ for } r > a$$

i.e. $$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\left[\frac{W^2r^2}{a^2}+l^2\right]R = 0 \quad ;\text{ for } r > a \qquad (5b)$$

where, $W=a\left(\beta^2-k_0^2n_2^2\right)^{1/2}$ (6b)

Now we define the normalized waveguide parameter V by,

$$V = \left(U^2+W^2\right)^{1/2} = \left\{a^2\left(k_0^2n_1^2-\beta^2\right)+a^2\left(\beta^2-k_0^2n_2^2\right)\right\}^{1/2}$$

$$= k_0a\left(n_1^2-n_2^2\right)^{1/2} = \frac{2\pi}{\lambda_0}a\left(NA\right) \qquad (7a)$$

Also $$\frac{W^2}{V^2} = \frac{a^2\left(\beta^2-k_0^2n_2^2\right)}{k_0^2a^2\left(n_1^2-n_2^2\right)} = \frac{\left(\beta^2-k_0^2n_2^2\right)}{k_0^2\left(n_1^2-n_2^2\right)} \qquad (7b)$$

Guided modes correspond to $k_0^2n_2^2<\beta^2<k_0^2n_1^2$ and therefore for guided modes both U and W are real.

Eqns.5a and 5b are of the form of standard Bessel's equations, [ $x^2y''+xy'+\left(x^2-\alpha^2\right)y = 0$ and $\tilde{x}^2y''+\tilde{x}y'-\left(\tilde{x}^2+\alpha^2\right)y = 0$] with $x = \frac{Ur}{a}$ and $\tilde{x}=\frac{Wr}{a}$.

The general solution of the first Bessel's equation is given by $AJ_l\left(x\right)+BY_l\left(x\right)$, where, $J_l\left(x\right)=\left(\frac{x}{2}\right)^l\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma\left(j+l+1\right)}\left(\frac{x}{2}\right)^{2j}$ and $Y_l\left(x\right)=\left(x\right)^{-l}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!2^{2j-l}\Gamma\left(j-l+1\right)}x^{2j}$. There are different forms of Bessel's function. Here we choose only one of them. In our case we reject the second part $BY_l\left(x\right)$ since $Y_l\left(x\right)$ diverges (tends to ∞) as x→0. The general solution of the second Bessel's equation is given by $AK_l\left(\tilde{x}\right)+BI_l\left(\tilde{x}\right)$, where, $K_l\left(\tilde{x}\right)$ and $I_l\left(\tilde{x}\right)$ are the modified Bessel's function. In the asymptotic form, they are given as,

$$K_l\left(\tilde{x}\right)_{\text{as } \tilde{x}\to\infty}\to\left(\frac{\pi}{2\tilde{x}}\right)^{1/2}e^{-\tilde{x}} \text{ and } I_l\left(\tilde{x}\right)_{\text{as } \tilde{x}\to\infty}\to\left(\frac{1}{2\pi\tilde{x}}\right)^{1/2}e^{\tilde{x}} \qquad (8)$$

Again we reject $I_l\left(\tilde{x}\right)$ in our case since $I_l\left(\tilde{x}\right)$ diverges (tends to ∞) as $\tilde{x}\to\infty$. Thus the solutions of eqns.5a and 5b respectively are $AJ_l\left(\frac{Ur}{a}\right)$ and $AK_l\left(\frac{Wr}{a}\right)$. By eqn.10 sec.3.15

$$\psi\left(r,\phi\right) = R\left(r\right)\Phi\left(\phi\right)$$

We assume that $\psi(r, \phi)$ and $\frac{\partial\psi}{\partial r}$ are continuous at the core-cladding interface (r = a). In order to satisfy these conditions we modify the solutions of eqns.5a and 5b, respectively, as $\frac{A}{J_l(U)} J_l\left(\frac{Ur}{a}\right)$ and $\frac{A}{K_l(W)} K_l\left(\frac{Wr}{a}\right)$. Also assuming $\Phi(\phi)$ is either a cosine function or a sine function of $\phi$, (refer eqn.4) we write,

$$\psi_1(r, \phi) = R_1(r)\Phi(\phi) = \frac{A}{J_l(U)} J_l\left(\frac{Ur}{a}\right)\begin{bmatrix}\cos(l\phi)\\ \sin(l\phi)\end{bmatrix}; \quad \text{for } r < a \tag{9a}$$

$$\psi_2(r, \phi) = R_2(r)\Phi(\phi) = \frac{A}{K_l(W)} K_l\left(\frac{Wr}{a}\right)\begin{bmatrix}\cos(l\phi)\\ \sin(l\phi)\end{bmatrix}; \quad \text{for } r > a \tag{9b}$$

We can easily show that when r = a eqn.9a and 9b reduce to a single equation showing that $\psi$ is continuous at the core-cladding interface,

$$\psi_1(r, \phi) = \psi_2(r, \phi) = A\begin{bmatrix}\cos(l\phi)\\ \sin(l\phi)\end{bmatrix} \tag{10}$$

Since $\frac{\partial\psi}{\partial r}$ is continuous at r = a,

$$\left(\frac{\partial\psi_1}{\partial r}\right)_{r=a} = \left(\frac{\partial\psi_2}{\partial r}\right)_{r=a}$$

i.e.
$$\left\{\frac{A}{J_l(U)}\frac{U}{a}J_l'\left(\frac{Ur}{a}\right)\begin{bmatrix}\cos(l\phi)\\ \sin(l\phi)\end{bmatrix}\right\}_{r=a} = \left\{\frac{A}{K_l(W)}\frac{W}{a}K_l'\left(\frac{Wr}{a}\right)\begin{bmatrix}\cos(l\phi)\\ \sin(l\phi)\end{bmatrix}\right\}_{r=a}$$

i.e.
$$\frac{UJ_l'(U)}{J_l(U)} = \frac{WK_l'(W)}{K_l(W)} \tag{11}$$

Now we use the two identities of the Bessel's function,

$$\frac{2\alpha}{x}J_\alpha(x) = J_{\alpha+1}(x) + J_{\alpha-1}(x)$$

And,
$$2\frac{d}{dx}\{J_\alpha(x)\} = J_{\alpha-1}(x) - J_{\alpha+1}(x)$$

In our case the identities become, (considering the functions and their derivatives at r = a),

$$\frac{2l}{U}J_l(U) = J_{l+1}(U) + J_{l-1}(U) \tag{12}$$

And
$$2J_l'(U) = J_{l-1}(U) - J_{l+1}(U) \tag{13}$$

Using eqn.12 in eqn.13 we obtain,

$$2J_l'(U) = \frac{2l}{U}J_l(U) - J_{l+1}(U) - J_{l+1}(U)$$

$$= \frac{2l}{U}J_l(U) - 2J_{l+1}(U)$$

i.e.
$$UJ_l'(U) = lJ_l(U) - UJ_{l+1}(U)$$

Or, $$\pm UJ_l'(U) = lJ_l(U) - UJ_{l\pm1}(U) \tag{14}$$

Similarly we can show that,

$$\pm WK_l'(W) = lK_l(W) \mp WK_{l\pm1}(W) \tag{15}$$

By eqn.11 $$\frac{UJ_l'(U)}{J_l(U)} = \frac{WK_l'(W)}{K_l(W)}$$

Using eqns.14 and 15 we get,

i.e. $$\frac{lJ_l(U)}{J_l(U)} - \frac{UJ_{l+1}(U)}{J_l(U)} = \frac{lK_l(W)}{K_l(W)} - \frac{WK_{l+1}(W)}{K_l(W)}$$

i.e. $$l - \frac{UJ_{l+1}(U)}{J_l(U)} = l - \frac{WK_{l+1}(W)}{K_l(W)}$$

i.e. $$\frac{UJ_{l+1}(U)}{J_l(U)} = \frac{WK_{l+1}(W)}{K_l(W)} \tag{16a}$$

Or eqn.11 can also be written as,

$$\frac{UJ_{l-1}(U)}{J_l(U)} = -\frac{WK_{l-1}(W)}{K_l(W)} \tag{16b}$$

However, using the proper limiting forms one can show that

$$\underset{W\to 0}{\text{Lt}}\left(\frac{WK_{l-1}(W)}{K_l(W)}\right) \to 0 \quad ; \text{ for } l = 0, 1, 2, .......... \tag{17}$$

Therefore we use eqn.16b for studying the modes. Also we have $J_{-1}(U) = -J_1(U)$ and $K_{-1}(W) = K_1(W)$. Then from eqn.16b for $l = 0$, we obtain,

$$\frac{UJ_1(U)}{J_0(U)} = \frac{WK_1(W)}{K_0(W)} \tag{18}$$

We would like to mention here that the boundary conditions used in deriving the eigenvalue equation (eqn.16b) are consistent with the approximation involved in using the scalar wave equation.

Now we define the normalized propagation constant as,

$$b = \frac{\frac{\beta^2}{k_0^2} - n_2^2}{n_1^2 - n_2^2} = \frac{\left(\beta^2 - k_0^2 n_2^2\right)}{k_0^2\left(n_1^2 - n_2^2\right)} = \frac{W^2}{V^2} \tag{19}$$

(Refer eqn.7).

Also we have, $$V^2 = U^2 + W^2$$

Using eqn.19, $$= U^2 + V^2 b$$

Thus, $$U^2 = V^2(1-b)$$

Or, $$U = V(1-b)^{1/2} \tag{20}$$

Then eqn.16b becomes,

$$\frac{UJ_{l-1}(U)}{J_l(U)} = -\frac{WK_{l-1}(W)}{K_l(W)} \tag{21}$$

$$\frac{V(1-b)^{1/2} J_{l-1}\left(V(1-b)^{1/2}\right)}{J_l\left(V(1-b)^{1/2}\right)} = -\frac{Vb^{1/2}K_{l-1}\left(Vb^{1/2}\right)}{K_l\left(Vb^{1/2}\right)} \quad ; \quad l \geq 1 \qquad (22)$$

And eqn.18 becomes,

$$\frac{UJ_1(U)}{J_0(U)} = \frac{WK_1(W)}{K_0(W)}$$

$$\frac{V(1-b)^{1/2} J_1\left(V(1-b)^{1/2}\right)}{J_0\left(V(1-b)^{1/2}\right)} = \frac{Vb^{1/2}K_1\left(Vb^{1/2}\right)}{K_0\left(Vb^{1/2}\right)} \quad ; \quad l = 0 \qquad (23)$$

For guided modes we must have,

$$k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2$$

i.e. $$n_2^2 < \frac{\beta^2}{k_0^2} < n_1^2$$

i.e. $$n_2^2 - n_2^2 < \frac{\beta^2}{k_0^2} - n_2^2 < n_1^2 - n_2^2$$

i.e $$0 < \frac{\beta^2}{k_0^2} - n_2^2 < n_1^2 - n_2^2$$

i.e. $$0 < \frac{\frac{\beta^2}{k_0^2} - n_2^2}{n_1^2 - n_2^2} < 1$$

i.e. $$0 < b < 1 \qquad (24)$$

For a given value of $l$, there will be a finite number of solutions for eqn.22 and the m[th] solution (m = 1, 2, 3, ....) is referred to as the $LP_{lm}$ mode. To find the modes we follow a graphical method. The values of LHS and RHS of eqn.22 are plotted against different values of b which satisfies eqn.24. The points of intersection of the two curves represent the discrete modes of the waveguides. It is clear that since $0 < b < 1$, there will be only a finite number of guided modes.

The guided modes (for a given $l$ value) which are given by the points of intersection are designated in decreasing values of 'b' as $LP_{l1}$, $LP_{l2}$, $LP_{l3}$, etc.

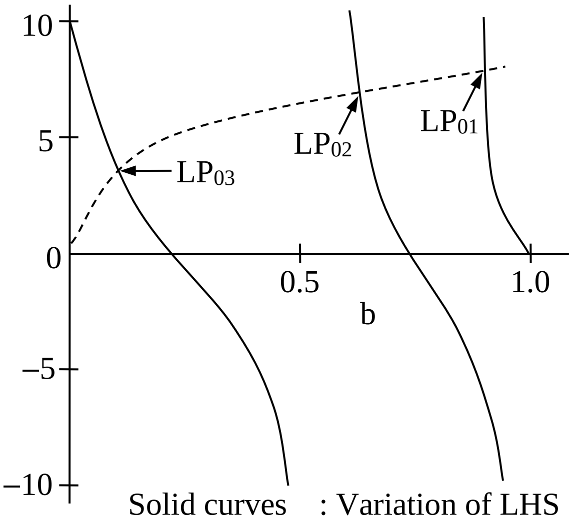



**Fig.a: for *l* = 0 and V = 8**

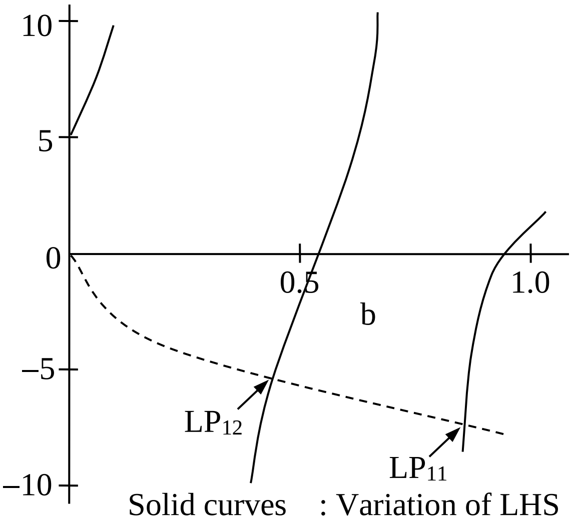



**Fig.b: for *l* = 1 and V = 8**

The variation of 'b' with 'V' forms a set of universal curves, which are plotted in fig.c. We can see from the fig.c that at a particular V value there are only a finite number of modes.

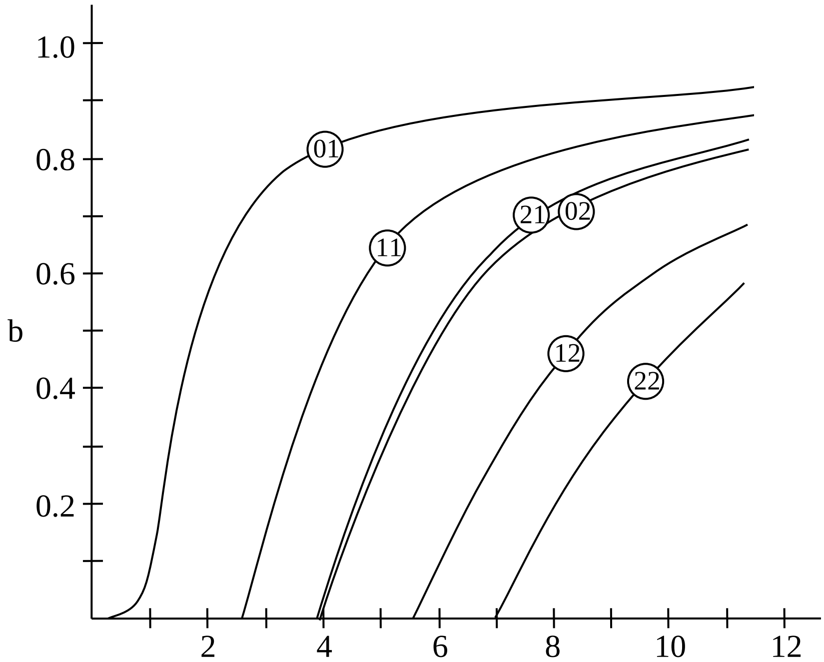



**Fig.c** V

**Table of cut off frequencies of various $LP_{lm}$ modes**

| *l* = 0 modes | $J_1(V_c) = 0$ |
|---|---|
| **Mode** | $V_c$ |
| $LP_{01}$ | 0 |
| $LP_{02}$ | 3.8317 |
| $LP_{03}$ | 7.0156 |
| $LP_{04}$ | 10.1735 |

| *l* = 1 modes | $J_0(V_c) = 0$ |
|---|---|
| **Mode** | $V_c$ |
| $LP_{11}$ | 2.4048 |
| $LP_{12}$ | 5.5201 |
| $LP_{13}$ | 8.6357 |
| $LP_{14}$ | 11.7915 |

| *l* = 2 modes | $J_1(V_c) = 0$; $V_c \neq 0$ |
|---|---|
| **Mode** | $V_c$ |
| $LP_{21}$ | 3.8317 |
| $LP_{22}$ | 7.0156 |
| $LP_{23}$ | 10.1735 |
| $LP_{24}$ | 13.3237 |

| *l* = 3 modes | $J_2(V_c) = 0$; $V_c \neq 0$ |
|---|---|
| **Mode** | $V_c$ |
| $LP_{31}$ | 5.1356 |
| $LP_{32}$ | 8.4172 |
| $LP_{33}$ | 11.6198 |
| $LP_{34}$ | 14.7960 |

The condition b = 0 (i.e. by eqn.19, $\beta^2 = k_0^2 n_2^2$) corresponds to what is known as the cut off of the mode. For b<0, $\beta^2 < k_0^2 n_2^2$ and the fields are oscillatory even in the cladding and the modes are known as radiation modes. At cut off, b = 0, by eqn.19, W = 0 and by eqn.20, U = V = $V_c$.

Then the cut offs of the various modes are determined from the following equations.

$l = 0$ modes; $J_1(V_c) = 0$ [by eqn.18 when W = 0]

$l = 1$ modes; $J_0(V_c) = 0$ [by eqn.16b when W = 0]

$l \geq 2$ modes; $J_{l-1}(V_c) = 0$ [by eqn.16b when W = 0]; $V_c \neq 0$

It should be noted that for $l \geq 2$, the root $V_c = 0$ must not be included since,

$$\underset{V\to 0}{\mathrm{Lt}}\left(\frac{VJ_{l-1}(V)}{J_l(V)}\right) \neq 0; \quad \text{for } l \geq 2 \tag{25}$$

Thus the cut off 'V' values, also known as the normalized cut off frequencies occur at the zeros of Bessel functions and are tabulated in the table given above.

By the above analysis and also from the graph shown in fig.c it is clear that for a step index fiber with,

$$0 < V < 2.4048 \tag{26}$$

there will be only one guided mode, namely $LP_{01}$ mode (refer the table also). Such a fiber is called a *single mode fiber*, which has tremendous importance in the optical fiber communication systems.

Fig.d represents a plot of the radial intensity distribution of some low order modes in a step index fiber for V = 8. Notice that higher modes have greater fraction of power in the cladding.

Fig.e represents the intensity distribution with respect to r and $\phi$ and the modal field patterns for some low order modes in a step index fiber.

Fig.f gives the modal intensity pattern of the $LP_{23,\,12}$ mode in a multimode fiber.

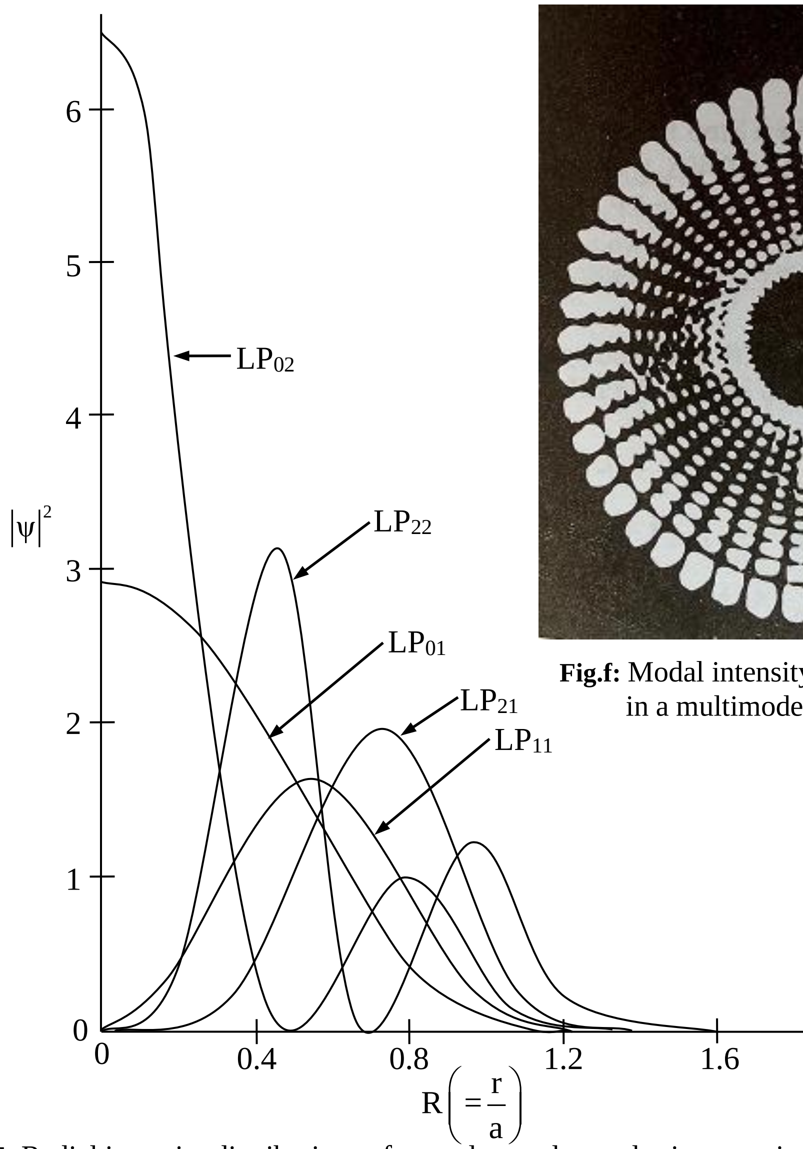



**Fig.d:** Radial intensity distributions of some low order modes in a step index fiber for V = 8

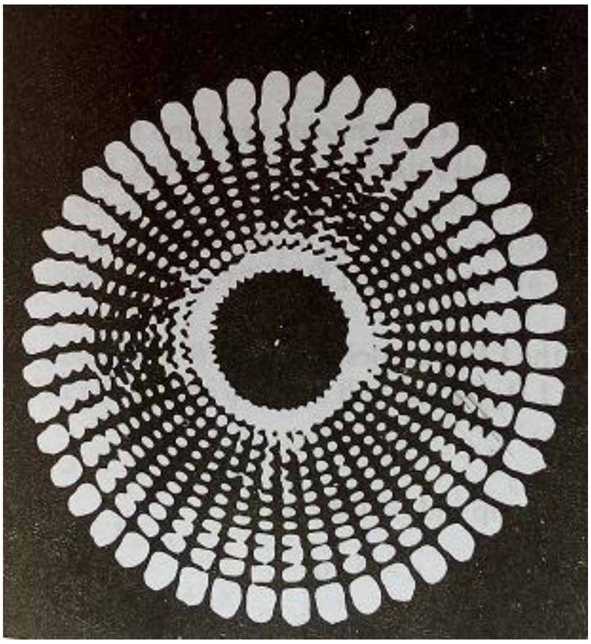


**Fig.f:** Modal intensity pattern of $LP_{23,\,12}$ mode in a multimode fiber

The following points are to be noted.

(a) The $l$ = 0 modes are twofold degenerate corresponding to two independent states of polarization.

(b) The $l \geq 1$ modes are four fold degenerate because for each polarization, the $\phi$ dependence could be either $\cos l\phi$ or $\sin l\phi$.

Further,

Number of zeros in the $\phi$ direction = $2l$ (27)

Number of zeros in the radial direction excluding that at r = 0 is = m − 1 (28)

Fig.f shows the mode filed distribution represents a typical higher order mode, say $LP_{23,\,12}$ mode, in a multimode fiber. When V>>1, the total number of modes in a step index multimode fiber is given by,

$$N \approx \frac{V^2}{2} \tag{29}$$

Such a fiber that can support large number of guided modes is known as a multimode fiber. For a typical multimode step index fiber, $n_1$ = 1.47, $n_2$ = 1.46, a = 25 μm and for $\lambda_0$ = 0.8μm, V ≈ 34. Such a fiber can support approximately 580 modes.

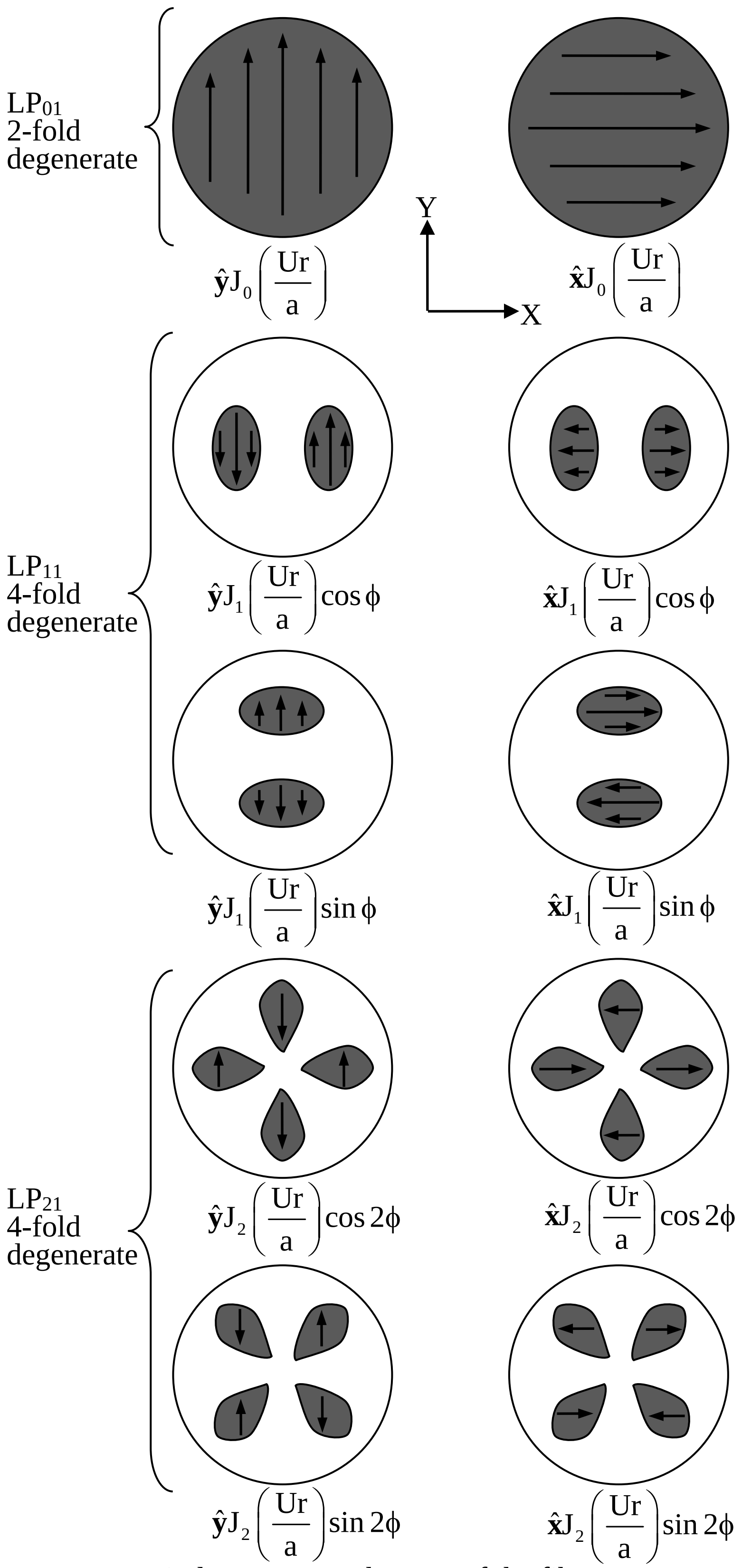



**Fig.e** Large circle represents the core of the fiber
Shaded region represents field distribution area
Arrows in the shaded region represents electric field direction

## 5.16.2 Fractional modal power in the core

One of the important parameters associated with the fiber optic waveguide is the fractional power carried in the core. The intensity of a wave is the energy flowing through unit area in one second, or it is the power per unit area. Thus

Power flowing through an area element dA = IdA (1)

Since the intensity is proportional to square of the amplitude of the wave, the power flowing through an area element dA of the core of the cylindrical optical fiber is given by,

$$dP_{core} \propto |\psi|^2 \, dA$$

$$\propto |\psi|^2 \, dr.rd\phi$$

∴ Total power flowing through the core of the cylindrical optical fiber is given by,

$$P_{core} \propto \int_0^a \int_0^{2\pi} |\psi|^2 \, rdrd\phi \qquad (2)$$

Using eqn.9a of sec.3.16.1, we obtain (constant A is included in the constant 2C and considering the cosine function)

$$P_{core} = \frac{2C}{J_l^2(U)} \int_0^a J_l^2\left(\frac{Ur}{a}\right) rdr \int_0^{2\pi} \cos^2(l\phi)\, d\phi \qquad (3)$$

$$= \frac{2\pi C}{J_l^2(U)} \int_0^a J_l^2\left(\frac{Ur}{a}\right) rdr \qquad (4)$$

Since, $\int_0^{2\pi} \cos^2(l\phi)\, d\phi = \pi$

Put $\frac{Ur}{a} = x$

$dr = \frac{a}{U} dx$

Then, $$P_{core} = \frac{2\pi C}{J_l^2(U)} \frac{a^2}{U^2} \int_0^U J_l^2(x)\, xdx \qquad (5)$$

[ $\int_0^U J_l^2(x)\, xdx$ is evaluated as follows. Let y = $J_l(x)$ is the solution of the Bessel's differential equation,

$$x^2 y'' + xy' + \left(x^2 - l^2\right) y = 0$$

Multiplying with $2y'$, we get,

$$2x^2 y' y'' + 2xy'^2 + \left(x^2 - l^2\right) 2yy' = 0$$

i.e. $$\frac{d}{dx}\left[x^2 y'^2 + x^2 y^2 - l^2 y^2\right] - 2xy^2 = 0$$

i.e. $$2xy^2 = \frac{d}{dx}\left[x^2 y'^2 + x^2 y^2 - l^2 y^2\right]$$

Then, $$\int_0^U 2xy^2 dx = x^2 y'^2 + x^2 y^2 - l^2 y^2$$

i.e. $$\int_0^U 2J_l^2(x)\,x\,dx = x^2 J_l'^2(x) + \left(x^2 - l^2\right) J_l^2(x)$$ ]

Then, $$P_{core} = \frac{2\pi C}{J_l^2(U)} \frac{a^2}{U^2} \int_0^U J_l^2(x)\,x\,dx = \frac{\pi C}{J_l^2(U)} \frac{a^2}{U^2} \left[ x^2 J_l'^2(x) + \left(x^2 - l^2\right) J_l^2(x) \right]_0^U$$

Using eqn.13and eqn.12

$$= \frac{C\pi a^2}{U^2 J_l^2(U)} \left[ x^2 \left\{ \frac{J_{l-1}(x) - J_{l+1}(x)}{2} \right\}^2 + \left(x^2 - l^2\right) x^2 \left\{ \frac{J_{l+1}(x) + J_{l-1}(x)}{2l} \right\}^2 \right]_0^U$$

$$= \frac{C\pi a^2}{U^2 J_l^2(U)} \left[ x^2 \left\{ \frac{J_{l-1}^2(x) + J_{l+1}^2(x) - 2J_{l-1}(x) J_{l+1}(x)}{4} \right\} + \right.$$

$$\left. \left(x^2 - l^2\right) x^2 \left\{ \frac{J_{l-1}^2(x) + J_{l+1}^2(x) + 2J_{l-1}(x) J_{l+1}(x)}{4l^2} \right\} \right]_0^U$$

$$= \frac{C\pi a^2}{U^2 J_l^2(U)} \left[ x^4 \left\{ \frac{J_{l-1}^2(x) + J_{l+1}^2(x) + 2J_{l-1}(x) J_{l+1}(x)}{4l^2} \right\} - x^2 \left\{ J_{l-1}(x) J_{l+1}(x) \right\} \right]_0^U$$

Using eqn12 in the first term

$$= \frac{C\pi a^2}{U^2 J_l^2(U)} \left[ x^2 J_l^2(x) - x^2 \left\{ J_{l-1}(x) J_{l+1}(x) \right\} \right]_0^U$$

$$= \frac{C\pi a^2}{U^2 J_l^2(U)} \left[ U^2 J_l^2(U) - U^2 \left\{ J_{l-1}(U) J_{l+1}(U) \right\} \right]$$ ]

$$= C\pi a^2 \left[ 1 - \frac{J_{l-1}(U) J_{l+1}(U)}{J_l^2(U)} \right]$$

Using eqns.16a and 16b,

$$P_{core} = C\pi a^2 \left[ 1 + \frac{W^2}{U^2} \frac{K_{l-1}(W) K_{l+1}(W)}{K_l^2(W)} \right] \tag{6}$$

Similarly the total power in the cladding

$$P_{cladding} = C\pi a^2 \left[ \frac{K_{l-1}(W) K_{l+1}(W)}{K_l^2(W)} - 1 \right] \tag{7}$$

Therefore, the total power in the fiber is given by,

$$P_{total} = P_{core} + P_{cladding} = C\pi a^2 \left[ \frac{K_{l-1}(W) K_{l+1}(W)}{K_l^2(W)} + \frac{W^2}{U^2} \frac{K_{l-1}(W) K_{l+1}(W)}{K_l^2(W)} \right]$$

$$= C\pi a^2 \left[ \frac{K_{l-1}(W) K_{l+1}(W)}{K_l^2(W)} \left( 1 + \frac{W^2}{U^2} \right) \right]$$

$$= C\pi a^2 \frac{V^2}{U^2} \left[ \frac{K_{l+1}(W) K_{l-1}(W)}{K_l^2(W)} \right] \tag{8}$$

(Refer eqn.7a or eqn.20)

Then the fractional power propagating through the fiber is given by,

$$\eta = \frac{P_{core}}{P_{total}} = \frac{C\pi a^2\left[1+\frac{W^2}{U^2}\frac{K_{l+1}(W)K_{l-1}(W)}{K_l^2(W)}\right]}{C\pi a^2\frac{V^2}{U^2}\left[\frac{K_{l+1}(W)K_{l-1}(W)}{K_l^2(W)}\right]} = \frac{U^2K_l^2(W)+W^2K_{l+1}(W)K_{l-1}(W)}{V^2\left[K_{l+1}(W)K_{l-1}(W)\right]}$$

$$= \frac{W^2}{V^2}+\frac{U^2}{V^2}\left\{\frac{K_l^2(W)}{K_{l+1}(W)K_{l-1}(W)}\right\} \qquad (9)$$

As the mode approaches cut off value, $W \to 0$, $U \to V_c$ and $V \to V_c$.

$$\text{Also } \underset{W\to 0}{K_0(W)} \to -\ln\left(\frac{W}{2}\right) \quad \text{and} \quad \underset{W\to 0}{K_l(W)} \to \frac{2^{l-1}(l-1)!}{W^l} = \frac{(l-1)!}{2}\left(\frac{2}{W}\right)^l \qquad (10)$$

$$\text{Then,} \quad \eta \to \begin{cases} 0 & \text{for } l=0 \text{ and } 1 \\ \left(\frac{l-1}{l}\right) & \text{for } l \geq 2 \end{cases} \qquad (11)$$

[When $l = 0$, $K_0(W) = \ln\left(\frac{2}{W}\right)$, then from eqn.9,

$$\eta_{co} = \frac{K_0^2(W)}{K_1(W)K_{-1}(W)} = \frac{K_0^2(W)}{K_1^2(W)}; \text{ since } K_1(W)=K_{-1}(W)$$

$$= \frac{\left\{\ln\left(\frac{2}{W}\right)\right\}^2}{\left(\frac{1}{W}\right)^2} = W^2\left\{\ln\left(\frac{2}{W}\right)\right\}^2 \to 0 \text{ as } W \to 0.$$

When $l = 1$, from eqn.9,

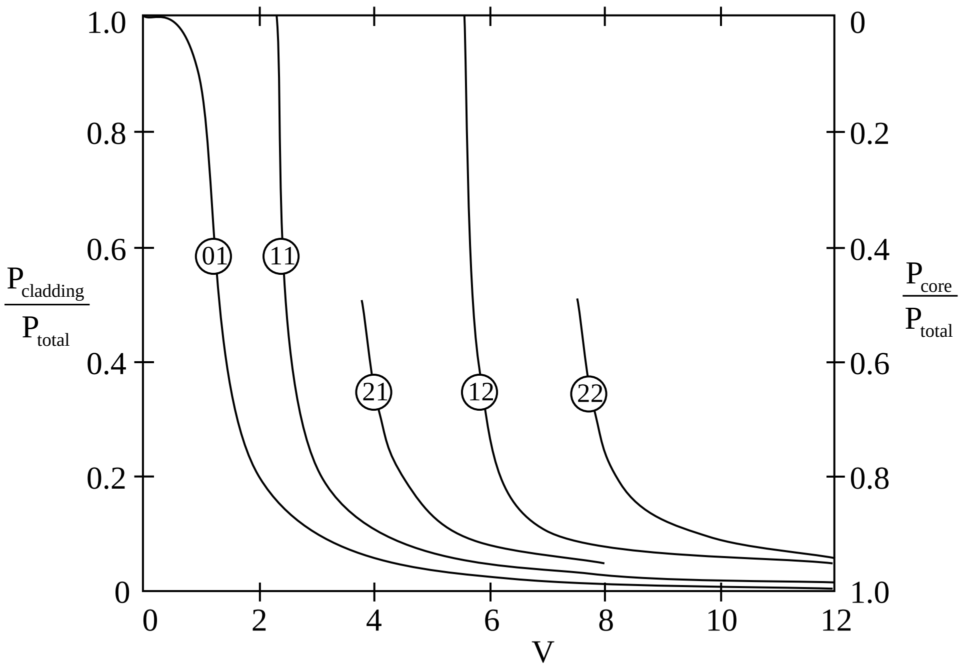
$$\eta_{co} = \frac{K_1^2(W)}{K_2(W)K_0(W)} = \frac{\frac{1}{W^2}}{\frac{2}{W^2}\ln\left(\frac{2}{W}\right)} = \frac{1}{2\ln\left(\frac{2}{W}\right)} \to 0 \text{ as } W \to 0.$$

When $l = l$, from eqn.9,

$$\eta_{co} = \frac{K_l^2(W)}{K_{l+1}(W)K_{l-1}(W)} = \frac{\frac{(l-1)!}{2}\frac{(l-1)!}{2}\left(\frac{2}{W}\right)^{2l}}{\frac{l!}{2}\left(\frac{2}{W}\right)^{l+1}\frac{(l-2)!}{2}\left(\frac{2}{W}\right)^{l-1}} = \frac{1}{l}(l-1)\frac{\left(\frac{2}{W}\right)^{2l}}{\left(\frac{2}{W}\right)^{2l}}$$

$$= \frac{l-1}{l} \text{ as } W \to 0. \qquad ]$$

Figure below gives the variation of fractional powers in the core and cladding plotted in the same graph as a function of V for the various modes of a step index fiber. From the graph it is clear that the power associated with a particular mode is concentrated in the core for large values of V, i.e. far from cut off.




## 5.17 Solved problems

1. A glass clad fiber is made with core glass of refractive index 1.55 and cladding is doped to give a refractive index 1.5. Calculate its numerical aperture the acceptance angle and the fractional index change.

Refractive index of core $n_1$ = 1.55

Refractive index of cladding $n_2$ = 1.50

Numerical aperture, N. A $= \sin\theta_0 = \sqrt{n_1^2 - n_2^2}$

$= \sqrt{1.55^2 - 1.5^2} = 0.3905$

Acceptance angle, $\theta_0 = \sin^{-1}\sqrt{n_1^2 - n_2^2}$

$= \sin^{-1} 0.3905 = 23°$

Fractional index change $\Delta = \dfrac{n_1 - n_2}{n_1}$

$= \dfrac{1.55 - 1.5}{1.55} = 0.03226$

2. Find the numerical aperture, acceptance angle and the critical angle of the fiber if light enters from air. Given refractive index of the core = 1.52 and the refractive index of the cladding = 1.47.

Refractive index of core $n_1$ = 1.52

Refractive index of cladding $n_2$ = 1.47

Numerical aperture, N. A $= \sin\theta_0 = \sqrt{n_1^2 - n_2^2}$

$$= \sqrt{1.52^2 - 1.47^2} = 0.38665$$

Acceptance angle, $\theta_0$ $= \text{Sin}^{-1}\sqrt{n_1^2 - n_2^2}$

$$= \text{Sin}^{-1} 0.38665 = 22°45'$$

$$\text{Sin}\theta_c = \frac{n_2}{n_1} = \frac{1.47}{1.52}$$

$$\therefore \quad \theta_c = 75°16'$$

3. A typical relative refractive index difference for an optical fiber designed for long distance transmission is 1%. Estimate the critical angle at the core-cladding interface, numerical aperture, the acceptance angle and the solid acceptance angle in air for the fiber when the core index is 1.46.

Relative refractive index difference

$$\Delta = \frac{n_1 - n_2}{n_1} = 1 - \frac{n_2}{n_1} = 0.01$$

$$\therefore \quad \frac{n_2}{n_1} = 1 - 0.01 = 0.99$$

Critical angle at the core-cladding interface

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}(0.99) = 81.89°$$

Numerical aperture $\approx n_1\sqrt{2\Delta} = 1.46 \times \sqrt{0.02} = 0.2065$

Acceptance angle, $\theta_0 = \sin^{-1}(N\,A) = \sin^{-1}(0.2065) = 11.92°$

Solid acceptance angle $= \pi\theta_0^2 \approx \pi \sin^2\theta_0$

$$= \pi\,(N\,A)^2 = \pi(0.2065)^2 = 0.1339 \text{ steradian}$$

4. Calculate the maximum radius for optic fiber with core of refractive index 1.515 and cladding of refractive index 1.495 if the fiber supports one mode at wavelength 1500 nm. [Knr. U April 2009]

Relative refractive index difference

$$\Delta = \frac{n_1 - n_2}{n_1} = \frac{1.515 - 1.495}{1.515} = 0.0132$$

For step index fiber for single mode propagation the maximum value of V is 2.4.

i.e. $V = 2.4$

But, $V = \frac{2\pi}{\lambda} a(N\,A) = \frac{2\pi}{\lambda} a n_1\sqrt{2\Delta}$

i.e. $\frac{2\pi}{\lambda} a n_1\sqrt{2\Delta} = 2.4$

Thus, the maximum radius for single mode propagation

$$a = \frac{2.4\lambda}{2\pi n_1\sqrt{2\Delta}} = \frac{2.4\times1500\times10^{-9}}{2\times3.14\times1.515\times\sqrt{2\times0.0132}} = 2.33\times10^{-6}\text{ m} = 2.33\ \mu\text{m}$$

5. A multimode step index fiber with a core diameter of 80 μm and a relative index difference of 1.5% is operating at a wavelength of 0.85 μm. If the core refractive index is 1.48, calculate the normalized frequency of the fiber and the number of guided modes.

$$V = \frac{2\pi}{\lambda}an_1\sqrt{2\Delta}$$

$$= \frac{2\times3.14\times40\times10^{-6}\times1.48\times\sqrt{2\times0.015}}{0.85\times10^{-6}} = 75.757$$

Number of guided modes $M_s \approx \frac{V^2}{2} = \frac{75.757^2}{2} = 2869$

6. A graded index fiber has a core with a parabolic refractive index profile which has a diameter of 60 μm. The fiber has a numerical aperture of 0.22. Estimate the total number of guided modes propagating in the fiber when it is operating at a wavelength of 1 μm.

$$V = \frac{2\pi}{\lambda}a(NA) = \frac{2\times3.14\times30\times10^{-6}\times0.22}{1\times10^{-6}} = 41.45$$

$$M_s \approx \left(\frac{\alpha}{\alpha+2}\right)\frac{V^2}{2}$$

For graded index fiber $\alpha = 2$

Then, $M_s \approx \frac{V^2}{4} = \frac{41.45^2}{4} = 429$

7. A 6 km optical link consists of multimode step index fiber with a core refractive index of 1.5 and a relative refractive index difference of 1%. Calculate the delay difference between the slowest and the fastest modes at the fiber output.

$$dT = \frac{Ln_1\Delta}{c} = \frac{6\times10^3\times1.5\times0.01}{3\times10^8} = 300\text{ ns}$$

## 5.18 Model questions

### Objective type questions

1. The basic principle of propagation of modulated signals through optical fiber is
   (A) Total internal reflection (B) Scattering of light
   (C) Refraction (D) Diffraction

2. Cladding material of an optical fiber has a refractive index
   (A) Greater than that of core material
   (B) Equal to that of core material
   (C) Lower than that of core material
   (D) None of these

3. The buffer and jacket layers
   (A) Contribute to the optical properties of the wave guide
   (B) Do not contribute to the optical properties of the wave guide (C) Contribute a little to the optical properties of the wave guide (D) None of these

4. The light gathering power of the fiber depends on

(A) Acceptance angle (B) Diameter of the core
(C) Cross sectional area of both core and cladding (D) None of these

5. The information carrying capacity of a fiber is greater when,
(A) the pulse dispersion is greater (B) the pulse dispersion is infinity (C) the pulse dispersion is smaller (D) None of these

6. The possible number of modes of an optical fiber depends on
(A) $\frac{d}{\lambda}$ (B) $\frac{\lambda}{d}$ (C) $\frac{d^2}{\lambda^2}$ (D) $\frac{\lambda^2}{d^2}$

**Short questions**

1. Describe an optical fiber. Give the dimensions of different parts.
2. What is the basic principle of fiber optic communication?
3. What is acceptance angle?
4. What is acceptance cone?
5. What is a step index fiber?
6. What is numerical aperture?
7. Explain dispersion in optical fibers.
8. What do you mean by step index fibers?
9. What is pulse dispersion? How does it affect the transmission through step index fibers?
10. What is a graded index fiber?
11. What is the advantage of graded index fiber over step index fiber?
12. Explain how the pulse dispersion is reduced in graded index fibers.
13. Distinguish between step index fibers and graded index fibers.
14. Explain the modes of propagation in optic fibers.
15. There is no pulse dispersion in single mode fibers. Why?
16. What are single mode and multimode fibers?
17. What is a plastic fiber?
18. What are the advantages of fiber optic communication?
19. Give the salient features of an optical fiber. Write some applications of optical fiber.
20. Briefly mention the uses of optical fibers.
21. Briefly explain the working of the optical fiber sensors.
22. What are the different types of optical fibers?
23. Explain the principle underlying the use of optical fibers in communication.
24. Explain why light passing through an optical fiber has very small energy loss.
25. What is an optical fiber? Give some applications.
26. Explain with a block diagram the use of optical fiber in communication.
27. Explain with a block diagram the principle and application of optical sensors. Mention the different types of optical sensors
28. How the optical fibers are used in military applications?
29. Explain the medical applications of optical fibers.
30. What is an optical fiber?
31. What are fiber characteristics?

**Short essay type questions**

1. Explain the terms ‘acceptance angle’, ‘acceptance cone’ and numerical aperture of optical fiber.
2. Derive the expression for the acceptance angle of the fiber.
3. What is a plastic fiber? Mention its advantages and disadvantages.

**Essay type questions**

1. Explain how light wave is propagated through a fiber. Derive the formula for numerical aperture. Show that it is approximately $n_1\sqrt{2\Delta}$.
2. Describe an optical fiber. Mention its applications.
3. What is an optical fiber? With the help of a block diagram explain how optical fibers are used for communication. Mention the advantages of this type of communication.
4. Explain the different applications of optical fibers.
5. Explain dispersion of optical fibers. What are its different types?
6. Write a note on the classification of optical fibers.
7. Write a note on fiber losses.

## 5.19 Problems

1. Calculate the numerical aperture and acceptance angle if the refractive index of the core is 1.48 and that of cladding is 1.46. [Ans. 0.2425, 14°]
2. Find the refractive indices of the core and the cladding materials of a fiber if numerical aperture is 0.22 and $\Delta = 0.012$. [Ans. $n_1 = 1.42$, $n_2 = 1.403$]
3. Assuming the outside medium is air, calculate the maximum value of angle of incidence that a ray can make with the axis of a step index fiber such that it gets guided through the fiber for the following fiber parameters (a) $n_1 = 1.6$, $n_2 = 1.5$, (b) $n_1 = 2.1$, $n_2 = 1.5$. [Ans. 33°50′, 90°]
4. A silica optical fiber has a core of refractive index 1.5 and cladding of refractive index 1.47. Determine (a) the critical angle at the core-cladding interface, (b) the N A for the fiber and (c) the acceptance angle in air for the fiber. [Ans. (a) 78.5°, (b) 0.30, (c) 17.4°]
5. A graded index fiber has a core with a parabolic refractive index profile which has a diameter of 50 μm. The fiber has a numerical aperture of 0.2. Calculate the total number of guided modes propagating in the fiber when it is operating at a wavelength of 1000 nm. [Ans. 247]
6. A step index fiber with a relative index difference of 1.5% is operating at a wavelength of 0.85 μm. If the core refractive index is 1.48, calculate the maximum core diameter for single mode operation of the fiber. [Ans. 2.534 μm]
7. A graded index fiber with a parabolic refractive index profile core has a refractive index at the core axis of 1.5 and a relative index difference of 1%. Estimate the maximum possible core diameter which allows single-mode operation at a wave length of 1300 nm. [Ans. 6.624 μm]
8. Determine the cut off wavelength for a step index fiber to exhibit single mode operation when the core refractive index and radius are 1.46 and 4.5 μm respectively, with the relative index difference being 0.25%. [Ans. 1213 nm]
9. An optical fiber has a numerical aperture of 0.2 and a cladding refractive index of 1.59. Determine the acceptance angle for the fiber in water which has a refractive index 1.33 and the critical angle at the core-cladding interface. [Ans.8.6°, 83.6°]
10. The mean optical power launched into an 8 km length of fiber is 120 μW and the mean optical power at the fiber output is 3 μW. Determine (a) the overall signal attenuation or loss in decibels through the fiber , (b) the signal attenuation per kilometre for the fiber, (c) the overall signal attenuation for a 10 km optical link using the same fiber with splices at 1 km intervals, each giving an attenuation of 1 dB and (d) the numerical input/output power ratio in (c). [Hint- Power ratio = $\frac{P_i}{P_o} = 10^{\frac{dB}{10}}$. Ans. (a) 16dB, (b) 2 dB/km, (c) 29 dB, (d) 794.3].

## 5.20 Pulse dispersion in single mode fibers

In the modal analysis of step index fiber we have seen that for single mode step index fiber the normalized V parameter is such that (eqn.26 sec.3.16.1)

$$0 < V < 2.4048 \tag{1}$$

That is, with this V parameter there is only one guided mode that can propagate through the fiber. Using equation for V eqn.1 becomes (by eqn.7a sec 3.16.1),

$$0 < k_0 a\left(n_1^2 - n_2^2\right)^{1/2} < 2.4048$$

$$0 < \frac{2\pi}{\lambda_0} a\left(n_1^2 - n_2^2\right)^{1/2} < 2.4048$$

$$0 < \frac{2\pi}{2.4048} a\left(n_1^2 - n_2^2\right)^{1/2} < \lambda_0$$

Thus, for a single mode step index fiber, the wavelength $\lambda_{co} = \frac{2\pi}{2.4048} a\left(n_1^2 - n_2^2\right)^{1/2}$ is called the cut off wavelength of the given step index fiber. Then we can write,

$$\lambda_0 > \lambda_{co} = \frac{2\pi}{2.4048} a\left(n_1^2 - n_2^2\right)^{1/2} \tag{2}$$

The fibers satisfying conditions given by eqn.1 or 2 are referred to as *single mode fibers* and they play very important role in high bandwidth optical fiber communication systems. Since there is only one mode they are free from intermodal dispersion, the major factor that limits the information capacity of multimode fibers.

The only form of dispersion in single mode fibers is the *intramodal dispersion* which is the broadening of a particular mode due to the finite spectral width of the source. It includes the material dispersion and the waveguide dispersion.

Material dispersion is caused by the dependence of refractive index of the material of the fiber on the wavelength of the propagating waves and the waveguide dispersion is the result of wavelength-dependence of the propagation constant of the optical waveguide. The phase velocity of the wave in a structure depends on its frequency simply due to the structure's geometry. The larger the wavelength, the more the fundamental mode will spread from the core into the cladding. This causes the fundamental mode to propagate faster. More generally, "waveguide" dispersion can occur for waves propagating through any inhomogeneous structure (e.g., a photonic crystal), whether or not the waves are confined to some region.

The dispersion is usually measured in picoseconds per kilometre length of the optical fiber per nanometre spectral width of the source. The material dispersion depends on the refractive index variation with wavelength and the spectral width of the source. (This is beyond the scope of the syllabus). It is given by,

$$\Delta\tau_m = -\frac{L}{c}\lambda_0^2 \frac{d^2 n_1}{d\lambda_0^2}\left(\frac{\Delta\lambda_0}{\lambda_0}\right) \tag{3a}$$

Or,
$$\frac{1}{L}\left(\frac{\Delta\tau_m}{\Delta\lambda_0}\right) = -\frac{\lambda_0}{c}\left(\frac{d^2 n_1}{d\lambda_0^2}\right) \text{ ps/km nm} \tag{3b}$$

where, $\Delta\tau_m$ is the pulse spread (in terms of time), L is the length of the waveguide, $n_1$ refractive index of the core and $\lambda_0$ is the free space wavelength. The suffix m in $\Delta\tau_m$ stands for material dispersion.

In a single mode step index fiber even if the material dispersion is absent there may be waveguide dispersion that can be calculated as follows.

We have, by eqn.7a sec.3.16.1, the normalized waveguide parameter V by,

$$V = \left(U^2 + W^2\right)^{1/2} = k_0 a\left(n_1^2 - n_2^2\right)^{1/2} = \frac{2\pi}{\lambda_0} a\left(n_1^2 - n_2^2\right)^{1/2} \quad (4)$$

$$\frac{dV}{d\lambda_0} = -\frac{2\pi}{\lambda_0^2} a\left(n_1^2 - n_2^2\right)^{1/2} = -\frac{V}{\lambda_0} \quad (5)$$

The variation of propagation constant $\beta$ of the wave in the fiber medium with respect to $\lambda_0$ is given by,

$$\frac{d\beta}{d\lambda_0} = \frac{d\beta}{dV}\frac{dV}{d\lambda_0} = -\frac{V}{\lambda_0}\frac{d\beta}{dV} \quad (6)$$

Here we have neglected the wavelength dependence of $n_1$ and $\Delta$ which is the fractional difference between the core and the cladding refractive indices given by

$$\Delta = \frac{n_1 - n_2}{n_1} \quad (7)$$

For weakly guiding fibers, $n_1 \approx n_2$, then we can write,

$$n_1 - n_2 = n_1\Delta \approx n_2\Delta \quad (8)$$

And, (by eqn.19 sec.3.16.1) the normalized propagation constant,

$$b = \frac{\frac{\beta^2}{k_0^2} - n_2^2}{n_1^2 - n_2^2} = \frac{\left(\frac{\beta}{k_0} + n_2\right)}{\left(n_1 + n_2\right)}\frac{\left(\frac{\beta}{k_0} - n_2\right)}{\left(n_1 - n_2\right)}$$

But propagation constant, $\beta = \frac{2\pi}{\lambda}$ and $k_0 = \frac{2\pi}{\lambda_0}$. Hence $\frac{\beta}{k_0} = \frac{\lambda_0}{\lambda} \approx n_1$. Also using eqn.8

Then, $$b \approx \frac{\frac{\beta}{k_0} - n_2}{n_2\Delta}$$

Or, $$\frac{\beta}{k_0} \approx bn_2\Delta + n_2$$

Thus, $$\beta \approx k_0 n_2\left(1 + b\Delta\right) = \frac{2\pi}{\lambda_0} n_2\left(1 + b\Delta\right) \quad (9)$$

$$\frac{d\beta}{dV} \approx 2\pi n_2\left(1 + b\Delta\right)\frac{d}{dV}\left(\frac{1}{\lambda_0}\right) + \frac{2\pi}{\lambda_0} n_2 \frac{d}{dV}\left(1 + b\Delta\right)$$

$$\approx 2\pi n_2\left(1 + b\Delta\right)\frac{d}{d\lambda_0}\left(\frac{1}{\lambda_0}\right)\frac{d\lambda_0}{dV} + \frac{2\pi}{\lambda_0} n_2 \frac{d}{dV}\left(1 + b\Delta\right)$$

$$\approx -\frac{2\pi n_2}{\lambda_0^2}\left(1 + b\Delta\right)\frac{d\lambda_0}{dV} + \frac{2\pi n_2}{\lambda_0}\Delta\frac{db}{dV} \quad (10)$$

The theory of optical fibers shows that the group delay in an optical fiber of length L,

$$\tau = L\frac{d\beta}{d\omega} = L\frac{d\beta}{d\left(\frac{2\pi c}{\lambda_0}\right)} = -\frac{L\lambda_0^2}{2\pi c}\frac{d\beta}{d\lambda_0} = -\frac{L\lambda_0^2}{2\pi c}\frac{d\beta}{dV}\frac{dV}{d\lambda_0}$$

$$\approx -\frac{L\lambda_0^2}{2\pi c}\left(-\frac{2\pi n_2}{\lambda_0^2}(1+b\Delta)\frac{d\lambda_0}{dV}+\frac{2\pi n_2}{\lambda_0}\Delta\frac{db}{dV}\right)\frac{dV}{d\lambda_0}$$

$$\approx \frac{L\lambda_0^2}{2\pi c}\frac{2\pi n_2}{\lambda_0^2}(1+b\Delta)\frac{d\lambda_0}{dV}\frac{dV}{d\lambda_0}-\frac{L\lambda_0^2}{2\pi c}\frac{2\pi n_2}{\lambda_0}\Delta\frac{db}{dV}\frac{dV}{d\lambda_0}$$

Using eqn.5,
$$\approx \frac{L\lambda_0^2}{2\pi c}\frac{2\pi n_2}{\lambda_0^2}(1+b\Delta)+\frac{L\lambda_0^2}{2\pi c}\frac{2\pi n_2}{\lambda_0}\Delta\frac{db}{dV}\frac{V}{\lambda_0}$$

i.e.
$$\tau \approx \frac{Ln_2}{c}\left(1+b\Delta+V\Delta\frac{db}{dV}\right) \tag{11}$$

The broadening of the pulse due to the waveguide dispersion is given by,

$$\Delta\tau_w = \left(\frac{d\tau}{d\lambda_0}\right)\Delta\lambda_0 \approx \left[\frac{d}{d\lambda_0}\left\{\frac{Ln_2}{c}\left(1+b\Delta+V\Delta\frac{db}{dV}\right)\right\}\right]\Delta\lambda_0$$

$$\approx \frac{Ln_2\Delta}{c}\left\{\frac{db}{d\lambda_0}+\frac{d}{d\lambda_0}\left(V\frac{db}{dV}\right)\right\}\Delta\lambda_0$$

$$\approx \frac{Ln_2\Delta}{c}\left\{\frac{db}{d\lambda_0}+\frac{dV}{d\lambda_0}\frac{db}{dV}+V\frac{d}{d\lambda_0}\left(\frac{db}{dV}\right)\right\}\Delta\lambda_0$$

$$\approx \frac{Ln_2\Delta}{c}\left\{\frac{db}{dV}\frac{dV}{d\lambda_0}+\frac{dV}{d\lambda_0}\frac{db}{dV}+V\frac{d}{dV}\left(\frac{db}{dV}\right)\frac{dV}{d\lambda_0}\right\}\Delta\lambda_0$$

Using eqn.5,
$$\approx \frac{Ln_2\Delta}{c}\left\{-2\frac{db}{dV}\frac{V}{\lambda_0}-V\frac{d}{dV}\left(\frac{db}{dV}\right)\frac{V}{\lambda_0}\right\}\Delta\lambda_0$$

$$\approx -\frac{Ln_2\Delta}{c}\left(\frac{\Delta\lambda_0}{\lambda_0}\right)V\left\{2\frac{db}{dV}+V\frac{d^2b}{dV^2}\right\}$$

$$\approx -\frac{Ln_2\Delta}{c}\left(\frac{\Delta\lambda_0}{\lambda_0}\right)V\frac{d^2(bV)}{dV^2} \tag{12}$$

We have already seen the universal curves (fig.c sec.3.16.1) for ‘b’ as a function of ‘V’ in the case of a step index fiber. Thus for step index fibers the quantity $V\frac{d^2(bV)}{dV^2}$ depends only on the v value. This is true only when $n_1 \approx n_2$ (since for the derivation of eqn.12 we have assumed $n_1 \approx n_2$). This is indeed the case for all practical fibers.

Now in order to get a numerical appreciation we use the following empirical relation by Rudolph and Neumann in 1976.

$$b = \left(A-\frac{B}{V}\right)^2 \quad ; \ 1.5 < V < 2.5 \quad ; \ A = 1.1428 \text{ and } B = 0.996 \tag{13}$$

This relation is accurate to only within 0.2% of the exact values. Even though the eqn.13 is not accurate we use it to get an idea of the comparison between the material dispersion and the waveguide dispersion.

By eqn.13 we have,

$$bV = V\left(A - \frac{B}{V}\right)^2$$

$$\frac{d(bV)}{dV} = \frac{d}{dV}\left\{V\left(A - \frac{B}{V}\right)^2\right\} = \left(A - \frac{B}{V}\right)^2 + 2\left(A - \frac{B}{V}\right)\frac{B}{V}$$

$$\frac{d^2(bV)}{dV^2} = \frac{d}{dV}\left\{\left(A - \frac{B}{V}\right)^2 + 2\left(\frac{AB}{V} - \frac{B^2}{V^2}\right)\right\} = 2\left(A - \frac{B}{V}\right)\frac{B}{V^2} - 2\frac{AB}{V^2} + 4\frac{B^2}{V^3}$$

$$= 2\frac{AB}{V^2} - 2\frac{B^2}{V^3} - 2\frac{AB}{V^2} + 4\frac{B^2}{V^3} = 2\frac{B^2}{V^3} \qquad (14)$$

Using eqn.14, eqn.12 becomes,

$$\Delta\tau_w \approx -\frac{Ln_2\Delta}{c}\left(\frac{\Delta\lambda_0}{\lambda_0}\right)V\frac{d^2(bV)}{dV^2} = -\frac{2Ln_2\Delta}{c}\left(\frac{\Delta\lambda_0}{\lambda_0}\right)\frac{B^2}{V^2}$$

Thus, $$\frac{1}{L}\left(\frac{\Delta\tau_w}{\Delta\lambda_0}\right) \approx -\frac{2n_2\Delta}{c\lambda_0}\frac{B^2}{V^2} \qquad (15)$$

But we have, by eqn.7a sec3.16.1

$$V = k_0 a\left(n_1^2 - n_2^2\right)^{1/2} = \frac{2\pi}{\lambda_0}a\left\{(n_1 + n_2)(n_1 - n_2)\right\}^{1/2}$$

Since, $n_1 \approx n_2$, $$V \approx \frac{2\pi}{\lambda_0}a\left\{(2n_2)(n_2\Delta)\right\}^{1/2}$$

$$\therefore \qquad V^2 \approx \frac{4\pi^2}{\lambda_0^2}a^2n_2^2(2\Delta) \qquad (16)$$

Then from eqn.15 we get,

$$\frac{1}{L}\left(\frac{\Delta\tau_w}{\Delta\lambda_0}\right) \approx -\frac{2n_2\Delta}{c\lambda_0}\frac{B^2}{\frac{4\pi^2}{\lambda_0^2}a^2n_2^2(2\Delta)} \approx -\frac{\lambda_0 B^2}{4\pi^2a^2n_2c} \qquad (17)$$

Eqn.17 shows that the relationship between $\frac{1}{L}\left(\frac{\Delta\tau_w}{\Delta\lambda_0}\right)$ and $\lambda_0$ is linear and the graph between them will be straight line with negative slope.

The total dispersion in a step index fiber is approximately,

$$\Delta\tau_{tot} \approx \Delta\tau_m + \Delta\tau_w \qquad (18)$$

We next study the relative contributions of the material and waveguide dispersions to the total dispersion. Figure below gives the variation of $\frac{1}{L}\left(\frac{\Delta\tau_m}{\Delta\lambda_0}\right)$, $\frac{1}{L}\left(\frac{\Delta\tau_w}{\Delta\lambda_0}\right)$ and $\frac{1}{L}\left(\frac{\Delta\tau_{tot}}{\Delta\lambda_0}\right)$ as a function of $\lambda_0$ for pure silica.

**Calculation of dispersions for typical cases**:

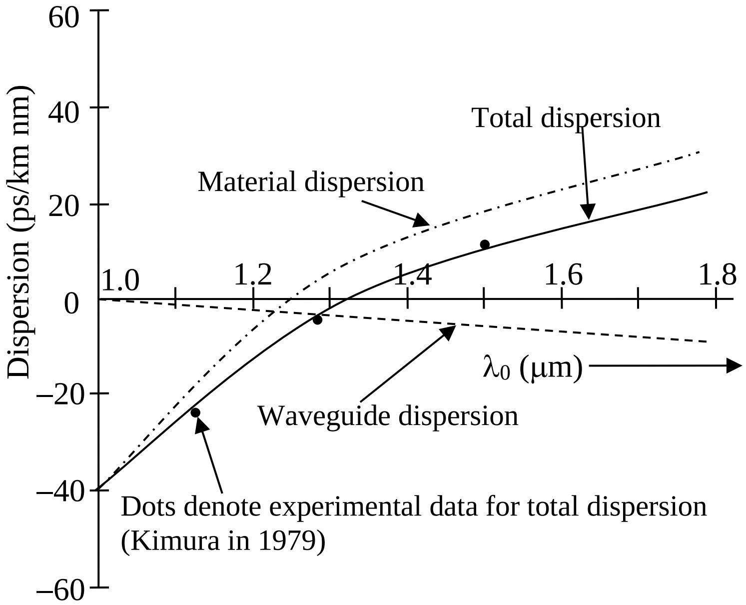
*Case-a: Material dispersion*: For $\lambda_0 = 0.8\mu m$; $\frac{d^2n_1}{d\lambda_0^2} \approx 4\times10^{-2}\ \mu m^{-2}$

Then, $$\frac{1}{L}\left(\frac{\Delta\tau_m}{\Delta\lambda_0}\right) = -\frac{\lambda_0}{c}\left(\frac{d^2n_1}{d\lambda_0^2}\right) = -\frac{0.8\times10^{-6}\times4\times10^{-2}\times10^{12}}{3\times10^8}$$

$$\approx -10^{-4}\ \text{s/m}^2 = -100\ \text{ps/km nm}$$

Since, $\frac{\text{ps}}{\text{km nm}} = \frac{10^{-12}\text{s}}{10^{3}\text{m}\times 10^{-9}\text{m}} = 10^{-6}\frac{\text{s}}{\text{m}^2}$; Or, $\frac{\text{s}}{\text{m}^2} = 10^{6}\frac{\text{ps}}{\text{km nm}}$

[The value –100 ps/km nm corresponding to $\lambda_0$ = 0.8μm is not plotted in the graph in which only values corresponding to $\lambda_0$ = 1μm to 1.8μm are plotted].




*Case-b: Waveguide dispersion*: For a step index single mode fiber with $\lambda_0$ = 0.8μm; a = 3μm; Δ = 0.00154; $n_2$ = 1.45. The values are chosen such that $V \simeq 1.9$ at $\lambda_0$ = 0.8μm. Using $B = 0.996$, we obtain,

$$\frac{1}{L}\left(\frac{\Delta\tau_w}{\Delta\lambda_0}\right) \approx -\frac{\lambda_0 B^2}{4\pi^2 a^2 n_2 c}$$

$$\approx -\frac{0.8\times10^{-6}\times0.996^2}{4\times3.14^2\times\left(3\times10^{-6}\right)^2\times1.45\times3\times10^8} \approx 5\times10^{-6}\ \text{m/s}^2 = -5\ \text{ps/km nm}$$

These results show that in the wavelength region around 0.8μm, the contribution due to material dispersion is much greater than that due to waveguide dispersion. Therefore the dispersion in optical fiber is mainly due to material dispersion.

*Case-c: Special case for $\lambda_0$ = 1.3μm*: It is evident from the graph that the material dispersion is very small and changes sign for the wavelength around $\lambda_0 \approx$ 1.3μm, so that the total dispersion is zero there. We can illustrate this by calculating the material and waveguide dispersions as done above.

For, $\lambda_0$ = 1.3μm; $\frac{d^2 n_1}{d\lambda_0^2} \approx -5.5\times10^{-4}\ \mu\text{m}^{-2}$ we get,

$$\frac{1}{L}\left(\frac{\Delta\tau_m}{\Delta\lambda_0}\right) \approx 2.4\ \text{ps/km nm}$$

For a step index single mode fiber with $\lambda_0$ = 1.3μm; a = 5.6μm; Δ = 0.00117; $n_2$ = 1.45. The values are chosen such that $V \simeq 1.9$ at $\lambda_0$ = 1.3μm. Using $B = 0.996$, we obtain,

$$\frac{1}{L}\left(\frac{\Delta\tau_w}{\Delta\lambda_0}\right) \approx -2.4\ \text{ps/km nm}$$

Thus, the material and waveguide dispersion cancel each other so that the total dispersion is zero and hence the fiber has a very large bandwidth. We mention here that the material dispersion calculated here is quite accurate, whereas the waveguide dispersion is not very accurate because of the use of the empirical formula given by eqn.13. Nevertheless, the above procedure tells us one may get zero dispersion in the fiber. (Complete the note and add more problems before print).